A DISCRETE GAUSS-GREEN IDENTITY FOR UNBOUNDED LAPLACE OPERATORS, AND THE TRANSIENCE OF RANDOM WALKS

PALLE E. T. JORGENSEN AND ERIN P. J. PEARSE

ABSTRACT. A resistance network is a connected graph (G, c). The conductance function c_{xy} weights the edges, which are then interpreted as resistors of possibly varying strengths. The relationship between the natural Dirichlet form $\mathcal{E}$ and the discrete Laplace operator Δ on a finite network is given by $\mathcal{E}(u, v) = \langle u, \Delta v \rangle_2$, where the latter is the usual ℓ^2 inner product. We extend this formula to infinite networks, where a new (boundary) term appears. The Laplace operator is typically unbounded in this context; we construct a reproducing kernel for the space of functions of finite energy which allows us to specify a dense domain for Δ and give several criteria for the transience of the random walk on the network. The extended Gauss-Green identity and the reproducing kernel also yield a boundary integral representation for harmonic functions of finite energy; this representation is the focus of a forthcoming paper.

CONTENTS

1. INTRODUCTION

Our purpose is to extend to infinite weighted graphs the Gauss-Green identity

$$\int_\Omega \nabla u \nabla v \, dV = - \int_\Omega u \Delta v \, dV + \int_{\partial\Omega} u \frac{\partial v}{\partial \mathbb{n}} \, dS, \qquad (1.1)$$

which expresses a fundamental relationship between analytical and geometrical notions of boundary. When the domain Ω is a network (connected weighted graph) G, then the boundary $\operatorname{bd} G = \partial\Omega$ will be a limit object similar to the Martin boundary of a Markov

chain (in particular, the boundary is not some prescribed subset of the domain). In this
discrete context, the left side of (1.1) corresponds to the (Dirichlet) energy form

$$\mathcal{E}(u, v) := \frac{1}{2} \sum_{x \in G^0} \sum_{y \in G^0} c_{xy}(u(x) - u(y))(v(x) - v(y)), \tag{1.2}$$

where G^0 is the vertex set of G. Furthermore, the Laplacian on the right side of (1.1) is
given pointwise by $(\Delta v)(x) := \sum_{y \sim x} c_{xy}(v(x) - v(y))$, where $x \sim y$ means that there is an
edge between x and y, and $c_{xy} = c_{yx} > 0$ is the weight ("conductance") of this edge. For
finite networks (or finite subnetworks of infinite networks), an elementary computation
shows that the energy $\mathcal{E}$ and Laplacian Δ are related by the formula

$$\mathcal{E}(u, v) = \sum_{G^0} u\Delta v, \tag{1.3}$$

so that the boundary term corresponding to $\int_{\partial\Omega} u\frac{\partial v}{\partial \mathrm{n}}\, dS$ in (1.1) vanishes. However, this
is not always the case for infinite networks. Let us define a function on G^0 to be *har-
monic* iff $\Delta u(x) = 0$ for every $x \in G^0$. Then it is clear that formula (1.3) cannot hold
for networks which support *nonconstant harmonic functions of finite energy* (also called
harmonic Dirichlet functions), because this would imply

$$0 < \mathcal{E}(h, h) = \sum_{x \in G^0} h(x)\Delta h(x) = \sum_{x \in G^0} h(x) \cdot 0 = 0, \tag{1.4}$$

where the strict inequality for nonconstant functions follows from the connectedness of the
network and the fact that (1.2) is a sum of nonnegative terms for $u = v = h$.

The main result of this paper is Theorem 3.12, in which we resolve (1.4) and describe
precise conditions for an infinite network to have a nonvanishing boundary term. In partic-
ular, with $\mathrm{dom}\,\mathcal{E} = \{u : \mathcal{E}(u) < \infty\}$, we have

$$\mathcal{E}(u, v) = \sum_{G^0} u\Delta v + \sum_{\mathrm{bd}\, G} u\frac{\partial v}{\partial \mathrm{n}}, \qquad \text{for all } u \in \mathrm{dom}\,\mathcal{E}, v \in \mathcal{M}, \tag{1.5}$$

where $\mathcal{M}$ is a certain domain for the Laplacian, and $\mathrm{bd}\, G$ and $\frac{\partial v}{\partial \mathrm{n}}$ are explained in Defini-
tion 3.9 and Definition 3.10. In particular, it should be noted that $\mathrm{bd}\, G$ *is **not** a subset of
G^0 or some larger ambient graph*; instead, this object may be understood as a collection
of certain equivalence classes of geodesics in the spirit of [Car72, Car73]. It is the focus
of [JP09a] to understand $\mathrm{bd}\, G$ as a measure space; see also [JP09d].

Until now, results in the literature on infinite networks have focused on finding suffi-
cient conditions to ensure the boundary term vanishes; see the discussion of the literature
just below. This paper also provides several conditions under which this occurs (see Re-
mark 3.16), but more importantly, it also provides conditions under which the boundary
term is *convergent and nonzero*. The correct determination of $\mathcal{M}$ is actually the primary
challenge for this; see Definition 3.4. We additionally give methods to compute the value
of such a boundary term.

Theorem 3.12 implies a boundary representation for harmonic functions of finite en-
ergy: Corollary 3.13 states that for a harmonic function u with $\mathcal{E}(u) < \infty$, one can recover
the values of u via

$$u(x) = \sum_{\text{bd } G} u \frac{\partial h_x}{\partial \mathfrak{n}} + u(o). \tag{1.6}$$

For now, we consider bd G and $\frac{\partial v}{\partial \mathfrak{n}}$ only as objects defined in terms of limits. However, Corollary 3.13 is the first step in the development of a boundary theory for harmonic functions of finite energy (akin to the Martin boundary theory for nonnegative harmonic functions, or the Poisson boundary theory for bounded harmonic functions). The paper [JP09a] formalizes the construction of bd G as a measure space (see 3.11). The resulting object bd G serves as a support set for a measure which represents the harmonic functions of finite energy, in the sense that

$$u(x) = \int_{\text{bd } G} \Bbbk(x, d\beta),$$

for any harmonic function u of finite energy, where $\Bbbk(x, d\beta)$ is a kernel whose second argument is a measure supported on bd G. Consequently, we expect that the *resistance boundary* bd G will be useful for studying induced Dirichlet forms on the boundary of an infinite resistance network. If φ is a function on bd G for which the harmonic extension $H\varphi$ to G has finite energy, then

$$\mathcal{E}_{\text{bd } G}(\varphi) := \mathcal{E}(H\varphi)$$

defines the induced Dirichlet form on the resistance boundary of G; see [Kig09a]. This provides an approach to doing analysis on spaces which are otherwise difficult to deal with, including self-similar fractals, Julia sets, and limit sets of Kleinian or Fuchsian groups. See [DS01, DS99, Kai03, Kig09a, LW09, JLW, RT10].

Remark 1.1 (Motivational examples). The boundary term does not vanish for homogeneous trees [Car72, Car73, Kig09a]; tesselations of the hyperbolic disk and graphs which are roughly isometric to hyperbolic spaces [LP09, §2 and §9]; graphs which are uniformly embedded in the hyprbolic disk [CW92]; and contacts graphs of circle packings in the plane [BS96]. Some examples are discussed in §6.

As an elementary example where the boundary term does not vanish, consider the following network discussed in Example 6.3: let G be the network whose vertices are the integers, where n is connected to $n - 1$ by an edge of conductance $c_{n,n-1} = c^n$ for some $c > 0$ and for $n > 0$, and symmetrically for $n < 0$:

$$\cdots \xrightarrow{c^3} \bullet_{-2} \xrightarrow{c^2} \bullet_{-1} \xrightarrow{c} \bullet_{0} \xrightarrow{c} \bullet_{1} \xrightarrow{c^2} \bullet_{2} \xrightarrow{c^3} \bullet_{3} \xrightarrow{c^4} \cdots \tag{1.7}$$

Lemma 6.5 gives a nonconstant harmonic function of finite energy h, and Remark 6.6 shows that for this function, the boundary term is $\sum_{\text{bd } G} h \frac{\partial h}{\partial \mathfrak{n}} = 1$.

For $c = 2$, this example is closely related to the example of the binary tree network (with $c_{xy} = 1$ for all $x, y \in G^0$). However, the network (1.7) provides an example where the Laplacian is unbounded and in fact is *not self-adjoint*. While the Laplacian as defined above is always a Hermitian operator, it is not even essentially self-adjoint in this example; that is, there exist multiple nontrivial self-adjoint extensions.

In [JP09e, §] and [JP09c, Ex. 13.39], we show that the network (1.7) supports a *defect vector*, that is, a function f satisfying $\mathcal{E}(f) < \infty$ and $\Delta f(x) = -f(x)$. (It turns out this function is also bounded.) For this function,

$$\sum_{x \in G^0} f(x) \Delta f(x) = \sum_{x \in G^0} f(x)(-f(x)) = -\sum_{x \in G^0} |f(x)|^2 \xrightarrow{n \to \infty} -\infty, \qquad (1.8)$$

and one would have a counterexample. This highlights the importance of $\mathcal{M}$; note that $f \notin \mathcal{M}$. Consequently, a key point of Theorem 3.12 is that for $v \in \mathcal{M}$, the two sums in (1.5) are both *finite*.

Notes on the literature. Random walks on graphs (and trees in particular) comprise an old and well-studied subject and we will not attempt to give complete references. However, we recommend [DS84, Doy88, LP09, LPW08] for introductory material and [Lyo83, Car73, Woe00], and the foundational paper [NW59] for more specific background. With regard to infinite graphs and finite-energy functions, see [Soa94, SW91, CW92, Dod06, PW90, PW88, Woe86, Tho90, BS96]. Effective resistance and resistance metric are studied extensively in [Kig03, Kig09b], and also in [Kig01].

For finite (sub) networks, a formula similar to (1.5) appears in [DK88, Prop 1.3], and another similar formula appears in [RSV97]. However, these authors do not give an extension of this formula to infinite networks.

Infinite networks are discussed in [KY89, Thm. 4.1], where the authors give some conditions under which (1.3) extends to infinite networks. However, these authors do not discuss the case when (1.3) must be replaced by (1.5). The emphasis of the present paper is on this latter case because the (nonvanishing) boundary term leads to a boundary representation for harmonic functions of finite energy, as well as detailed information about the structure of the space of functions of finite energy, and several computational techniques discussed below. Furthermore, the conditions provided by Kayano and Yamasaki may be highly nontrivial to verify. By contrast, we develop a dense subspace of functions to which the formula may be applies, and in [JP09b], we show that these functions are relatively easy to compute.

A reproducing kernel similar to energy kernel $\{v_x\}_{x \in G^0}$ constructed in §2.1 appears in [MYY94] under the name *Kuramochi kernel*. Indeed, the Kuramochi kernel can be recovered as a special case of the present work. As an advantage of the present approach, we note that our formulation puts the Green function in the same space as the reproducing kernel elements v_x. Also, the geometry of the Hilbert space studied in this paper is much simpler, in comparison to that induced by the inner product studied in [MYY94]; see §4.1.

It is easy to think of the formula (1.3) as $\mathcal{E}(u,v) = \langle u, \Delta v \rangle_{\ell^2}$, and hence the reader may be tempted to see our results as part of Kato's theory. Recall that [Kat95] gives a bijective correspondence between the family of closed symmetric quadratic forms $\mathcal{E}$ on a Hilbert space $\mathcal{H}$ and the family of nonnegative definite self-adjoint operators A on $\mathcal{H}$ via

$$\text{dom}\,\mathcal{E} = \text{dom}(\sqrt{A}), \quad \mathcal{E}(u,v) = \left\langle \sqrt{A}u, \sqrt{A}v \right\rangle_{\mathcal{H}}.$$

See [FŌT94, Thm. 1.3.1] and its corollary. However, if one requires $u, v \in \ell^2(G^0)$, then one does not usually obtain a dense subspace of the functions of finite energy. On the other hand, requiring that u and v have finite energy does not guarantee that either is in $\ell^2(G^0)$. Furthermore, even if Kato's theory could be adapted to accommodate this issue of domains, it still does not account for the second term on the right side of (1.5).

The reader may also notice a similarity between the Hilbert space $\mathcal{H}_{\mathcal{E}}$ which is the focus of the present study, and the extended Dirichlet space $\mathcal{F}_e$ studied in [FŌT94, §1.5–§1.6] and elsewhere. However, the resemblance is only superficial and in general there is no relation between the two. Note that $\mathcal{F}_e$ is a Hilbert space if and only if the Dirichlet form

is transient, and that the ℓ^2 functions are dense in $\mathcal{F}_e$ [FŌT94, Thm. 1.5.3]. These are both false for $\mathcal{H}_{\mathcal{E}}$.

Applications to mathematical physics. Due to the discrete nature of (1.5), this formula has applications beyond the problems of geometry. In particular, there is an immediate interpretation in terms of functions on $X \times X$ (where X is a discrete set) which is naturally understood in terms of matrices; see [JP09e].

The symmetric form of the identity (1.1) is

$$\int_\Omega (u\Delta v - v\Delta u)\, dV = \int_{\partial\Omega} (u\tfrac{\partial v}{\partial \mathbf{n}} - v\tfrac{\partial u}{\partial \mathbf{n}})\, dS,$$

and in a Hilbert space, this takes the form

$$\left(\langle \Delta^* u, v \rangle - \langle u, \Delta^* v \rangle \right) = \beta(u, v), \qquad u, v \in \operatorname{dom} \Delta^*, \tag{1.9}$$

where Δ^* is the adjoint of Δ with respect to the inner product of $\mathcal{H}$, and the right-hand side of (1.9) defines a *boundary form* in the sense of [DS88]. Many problems for transformations in linear algebra may be formulated with the use of symmetric matrices and the determination of β in (1.9) is an immediate question. In finite dimensions, all inner products are equivalent, and the boundary term is trivial. However, this is not so for problems from statistics, particle models, or infinite weighted graphs which require analysis in infinite dimensions [ADV09, Han96]. These analyses entail a choice of inner product, and therefore by completion, a Hilbert space $\mathcal{H}$, and rigorous study of boundary problems for an infinite discrete system X typically dictates a particular Hilbert space. If X is the vertex set of an infinite graph, it is tempting to take $\mathcal{H} = \ell^2(X)$ as the preferred Hilbert space, but it turns out that $\ell^2(X)$ gives a trivial boundary theory for X, in the sense of (1.5).

In this context, results of the present paper may be applied to certain discrete analogues of Laplace (or Schrödinger) operators. However, our framework applies to a variety of other discrete models as well, for example to operators in statistical physics generating the dynamics of particle spin-flips; see e.g., [CL07, FHM03, Lig78]. For example, consider the following close relative of the Laplace operator which plays a key role in spin dynamics:

$$(L_{ISF} f)(\eta) = \sum_{x \in \Lambda} c(x, \eta)(f(\eta^x) - f(\eta)), \qquad \eta \in \{0, 1\}^\Lambda$$

where η is a *configuration* of particle spins,

$$\eta^x(y) = \begin{cases} 1 - \eta(x), & y = x, \\ \eta(y), & y \neq x, \end{cases}$$

and $c(x, \eta)$ is a nonnegative function translation-invariant which is invariant on the lattice Λ. The case $\Lambda = \mathbb{Z}$ is discussed in [Lig78].

Outline. The rest of this paper is as follows:

§2 develops the Hilbert space structure of the functions of finite energy, in terms of a certain reproducing kernel $\{v_x\}$ for $\operatorname{dom}\mathcal{E}$. This kernel is necessary for formulating the hypotheses of Theorem 3.12, but it also gives insight into the Hilbert space structure of $\operatorname{dom}\mathcal{E}$. Additionally, this kernel (and the associated framework developed in the present paper) is used to study effective resistance metric on infinite networks in the forthcoming paper [JP09b], to construct a new boundary integral representation for harmonic functions

of finite energy in the forthcoming paper [JP09a], and to obtain information about the   1
spectrum of the Laplacian in this context.   2

§3 contains the complete statement and proof of Theorem 3.12, as well as the precise   3
definitions required. In particular, we discuss the class of *monopoles*, which can be thought   4
of as an extension of Green's kernel, and their relationship to the reproducing kernel(s)   5
introduced in the previous section. Corollary 3.13 introduces a boundary representation   6
for harmonic functions studied further in [JP09a].   7

§4 relates the boundary term $\sum_{\text{bd } G} u \frac{\partial v}{\partial \mathbb{n}}$ to the transience of the network. We indicate   8
how the characterizations of transience due to [Lyo83] and [NW59] appear in the present   9
context, and introduce three new equivalent characterizations of transience, stated in terms   10
of our main formula Theorem 3.12 and the operator-theoretic properties of Δ.   11

§5 concerns implications and applications of (1.5), and some facts about useful special   12
cases. We recover several known facts about finitely-supported functions and harmonic   13
functions; the use of the energy kernel $\{v_x\}$ and other techniques introduced in previous   14
sections allow for shorter and easier proofs. We also give a result indicating how one can 15
extend Theorem 3.12 to networks which contain vertices of infinite degree.   16

§6 gives a family of examples that illustrates the properties we discuss, and several   17
of our results. We prove that these simple networks support monopoles and harmonic   18
functions, and discuss why Δ may not be self-adjoint on these networks for appropriate   19
choices of c.   20

1.1. **Basic terms.** We now proceed to introduce the key notions used throughout this pa- 21
per: resistance networks, the energy form $\mathcal{E}$, the Laplace operator Δ, and the elementary   22
relations amongst them.   23

Definition 1.2. An resistance network is a connected graph (G, c), where G is a graph   24
with vertex set G^0, and c is the *conductance function* which defines adjacency by $x \sim y$ 25
iff $c_{xy} > 0$, for $x, y \in G^0$. We assume $c_{xy} = c_{yx} \in [0, \infty)$, and write $c(x) := \sum_{y \sim x} c_{xy}$. We   26
require $c(x) < \infty$, but $c(x)$ need not be a bounded function on G^0. The notation c may be   27
used to indicate the multiplication operator $(cv)(x) := c(x)v(x)$.   28

In this definition, *connected* means simply that for any $x, y \in G^0$, there is a finite se-   29
quence $\{x_i\}_{i=0}^n$ with $x = x_0$, $y = x_n$, and $c_{x_{i-1}x_i} > 0$, $i = 1, \ldots, n$. Conductance is the   30
reciprocal of resistance, so one can think of (G, c) as a network of nodes G^0 connected by   31
resistors of resistance c_{xy}^{-1}. We may assume there is at most one edge from x to y, as two   32
conductors c_{xy}^1 and c_{xy}^2 connected in parallel can be replaced by a single conductor with   33
conductance $c_{xy} = c_{xy}^1 + c_{xy}^2$. Also, we assume $c_{xx} = 0$ so that no vertex has a loop, as   34
electric current would never flow along a conductor connecting a node to itself.[1]   35

Definition 1.3. The *Laplacian* on G is the linear difference operator which acts on a func-   36
tion $v : G^0 \to \mathbb{R}$ by   37

$$(\Delta v)(x) := \sum_{y \sim x} c_{xy}(v(x) - v(y)). \qquad (1.10)$$

A function $v : G^0 \to \mathbb{R}$ is called *harmonic* iff $\Delta v \equiv 0$. The Laplacian may be written   38
$\Delta = c - \mathrm{T}$, where $(\mathrm{T} v)(x) := \sum_{y \sim x} c_{xy}v(y)$ is the *transfer operator*.   39

---

[1]Nonetheless, self-loops may be useful for technical considerations: one can remove the periodicity of a
random walk by allowing self-loops. This can allow one to obtain a "lazy walk" which is ergodic, and hence
more tractable. See, for example, [LPW08, LP09].

The network Laplacian defined above should not be confused with the normalized Laplace operator defined by $c^{-1}\Delta$, which appears frequently in the probability literature (e.g. [DS84]), nor the version $c^{-1/2}\Delta c^{-1/2}$ which appears in the literature on spectral graph theory (e.g., [Chu01]). Also note that we have adopted the physics convention (so that the spectrum is nonnegative) and thus our Laplacian is the negative of the one commonly found in the PDE literature.

Remark 1.4. The reader may wonder why we have elected to define the Laplacian as above, instead of one of the other mentioned options. A primary reason for our use of the unnormalized Laplacian is that it may be unbounded, and therefore have a delicate spectral structure which is not visible in the bounded case. In particular, the Laplacian given by (1.10) may fail to be *essentially self-adjoint*. This means that Δ has multiple self-adjoint extensions; or equivalently, nontrivial defect spaces. The issue of essential self-adjointness is closely related to the question of whether or not the graph is stochastically completeness, and in some contexts the two are equivalent. R. Wojciechowski has recently given a characterization of stochastic completeness for certain classes of graphs; see [Woj07, Woj09, Woj10]. Also, M. Keller and D. Lenz characterize stochastic completeness for regular Dirichlet forms on discrete sets in [KL10], and relate this to the stochastic completeness of graphs and subgraphs. Additionally, A. Weber has recently given a condition for stochastic completeness of graphs which may be understood as a weak curvature bound [Web08]. See [JP09e] for further definitions and discussion. We expect the results of the present paper to be useful for studying stochastic completeness.

Definition 1.5. An *exhaustion* of G is an increasing sequence of finite and connected subgraphs $\{G_k\}$, so that $G_k \subseteq G_{k+1}$ and $G = \bigcup G_k$.

Definition 1.6. The notation

$$\sum_{x \in G^0} := \lim_{k \to \infty} \sum_{x \in G_k} \tag{1.11}$$

is used whenever the limit is independent of the choice of exhaustion $\{G_k\}$ of G. This is clearly justified, for example, whenever the sum has only finitely many nonzero terms, or is absolutely convergent as in the definition of $\mathcal{E}$ just below.

Definition 1.7. The *energy* of functions $u, v : G^0 \to \mathbb{R}$ is given by the (closed, bilinear) Dirichlet form

$$\mathcal{E}(u, v) := \frac{1}{2} \sum_{x \in G^0} \sum_{y \in G^0} c_{xy}(u(x) - u(y))(v(x) - v(y)), \tag{1.12}$$

with the energy of u given by $\mathcal{E}(u) := \mathcal{E}(u, u)$. The *domain of the energy* is

$$\operatorname{dom} \mathcal{E} = \{u : G^0 \to \mathbb{R} : \mathcal{E}(u) < \infty\}. \tag{1.13}$$

Since $c_{xy} = c_{yx}$ and $c_{xy} = 0$ for vertices which are not adjacent, there is exactly one term in the sum in (1.12) for each edge in the network when we include the factor of $\frac{1}{2}$ to prevent double-counting. Note that (1.12) implies

$$\mathcal{E}(\delta_x) = c(x), \qquad \text{and} \qquad \mathcal{E}(\delta_x, \delta_y) = -c_{xy}, \tag{1.14}$$

where δ_x is a (unit) Dirac mass at $x \in G^0$. Consequently, the Dirac masses δ_x and δ_y are orthogonal with respect to energy if and only if $x \not\sim y$.

The following proposition may be found in [Str06, §1.3] or [Kig01, Ch. 2], for example.   1

Proposition 1.8. *The following properties are readily verified:*   2

(1) $\ker \mathcal{E}$ *consists precisely of the constant functions.*   3
(2) *(Polarization)* $\mathcal{E}(u, v) = \frac{1}{4}[\mathcal{E}(u + v) - \mathcal{E}(u - v)]$.   4
(3) *(Markov property)* $\mathcal{E}([u]) \leq \mathcal{E}(u)$, *where* $[u]$ *is any contraction[2] of* u.   5

Proposition 1.9. *Let* G^0 *be finite. Then*   6

$$\mathcal{E}(u, v) = \sum_{x \in G^0} u(x)\Delta v(x), \tag{1.15}$$

and all harmonic functions of finite energy are constant.   7

Proof. The computation is elementary, but we include it for later reference: 8

$$\sum_{x \in G^0} \overline{u}(x)\Delta v(x) = \frac{1}{2}\sum_{x \in G^0} \overline{u}(x)\Delta v(x) + \frac{1}{2}\sum_{y \in G^0} \overline{u}(y)\Delta v(y)$$

$$= \frac{1}{2}\sum_{x \in G^0}\sum_{y \sim x} c_{xy}\overline{u}(x)(v(x) - v(y)) - \frac{1}{2}\sum_{y \in G^0}\sum_{x \sim y} c_{xy}\overline{u}(y)(v(x) - v(y))$$

$$= \frac{1}{2}\sum_{x \in G^0}\sum_{y \sim x} c_{xy}(\overline{u}(x) - \overline{u}(y))(v(x) - v(y)),$$

If h is harmonic, then apply this formula to get $\mathcal{E}(h) = \sum_{x \in G^0} h(x)\Delta h(x) = 0$, and the latter   9
result follows from (1) of Proposition 1.8.   □   10

Connectedness is implicit in the calculations behind Proposition 1.8 and Proposition 1.9;   11
recall that *all* resistance networks considered in this work are connected. Our main result   12
is an extension of the above formula for $\mathcal{E}(u, v)$ to infinite networks in Theorem 3.12,   13
where the formula is more complicated: (1.5) indicates the presence of a "boundary term"   14
$\sum_{\text{bd } G} u\frac{\partial v}{\partial \mathbf{n}}$. It is shown in Theorem 4.6 that the presence of the boundary term corresponds   15
to the transience of the random walk on the underlying network, that is, the Markov process   16
with countable state space G^0 and transition probabilities $p(x, y) := c_{xy}/c(x)$.   17
In the traditional study of Dirichlet forms [FŌT94] or more general quadratic forms   18
[Kat95], one would write $\mathcal{E}(u, v) = \langle u, \Delta v\rangle_{\ell^2}$ and consider the space of functions with   19
$\|u\|_2 + \mathcal{E}(u) < \infty$. In classical potential theory (or Sobolev theory), this would amount to   20
working with the class of functions satisfying $\|f'\|_2 < \infty$, but abandoning the requirement   21
that $\|f\|_2 < \infty$. In our context, this is counterproductive: many of the most interesting   22
functions in dom $\mathcal{E}$ are not in $\ell^2(G^0)$; see Corollary 5.6. Consider that if at least two   23
connected components of $G\backslash\{o\}$ are infinite, then $v_x \notin \ell^2$ for vertices x in these components,   24
where v_x is an element of the energy kernel; see Definition 2.6. Also, in consideration of   25
(1.8), it is impossible for a defect vector of Δ to be in ℓ^2.   26

2. THE ENERGY HILBERT SPACE   27

In this section, we study the Hilbert space $\mathcal{H}_{\mathcal{E}}$ of (finite-energy) voltage functions, that   28
is, equivalence classes of functions $u : G^0 \to \mathbb{C}$ where $u \simeq v$ iff $u - v$ is constant. On this   29
space, the energy form is an inner product, and there is a natural reproducing kernel $\{v_x\}_{x \in G^0}$ 30

---

[2]In this context, a *contraction* is any operation $u \mapsto [u]$ such that $|[u](x) - [u](y)| \leq |u(x) - u(y)|$ whenever $x \sim y$. For example, let $[u] := \min\{1, \max\{0, u\}\}$.

indexed by the vertices; see Corollary 2.7. Since we work with respect to the equivalence relation defined just above, most formulas are given with respect to differences of function values; in particular, the reproducing kernel is given in terms of differences with respect to some chosen "origin". Therefore, for any given resistance network, we fix a reference vertex $o \in G^0$ to act as an origin. It will readily be seen that all results are independent of this choice. When working with representatives, we typically abuse notation and use u to denote the equivalence class of u. One natural choice is to take u so that $u(o) = 0$; a different but no less useful choice is to pick k so that $v = 0$ outside a finite set as discussed further in Definition 2.10.

Let $\mathbf{1}$ denote the constant function with value 1 and recall that $\ker \mathcal{E} = \mathbb{C}\mathbf{1}$.

Definition 2.1. The energy form $\mathcal{E}$ is symmetric and positive definite on $\mathrm{dom}\,\mathcal{E}$. Then $\mathrm{dom}\,\mathcal{E}/\mathbb{C}\mathbf{1}$ is a vector space with inner product and corresponding norm given by

$$\langle u, v \rangle_{\mathcal{E}} := \mathcal{E}(u, v) \quad \text{and} \quad \|u\|_{\mathcal{E}} := \mathcal{E}(u, u)^{1/2}. \tag{2.1}$$

The *energy Hilbert space* $\mathcal{H}_{\mathcal{E}}$ is the completion of $\mathrm{dom}\,\mathcal{E}/\mathbb{C}\mathbf{1}$ with respect to (2.1).

It can be checked directly that the above completion consists of (equivalence classes of) functions on G^0 via an isometric embedding into a larger Hilbert space as in [LP09, MYY94] or by a standard Fatou's lemma argument as in [Soa94].

Remark 2.2 (Four warnings about $\mathcal{H}_{\mathcal{E}}$).

(1) $\mathcal{H}_{\mathcal{E}}$ has no canonical o.n.b.; the usual candidates $\{\delta_x\}$ are not orthogonal by (1.14), and typically their span is not even dense; cf. Corollary 2.17.
(2) Pointwise identities should not be confused with Hilbert space identities; see Remark 2.13 and Lemma 3.7.
(3) Multiplication operators are not generally Hermitian, as we show in the next lemma contrasts sharply with more familiar Hilbert spaces.
(4) There is no natural interpretation of $\mathcal{H}_{\mathcal{E}}$ as an ℓ^2-space of functions on the vertices G^0 or edges G^1 of (G, c).

With regard to (4), it should be noted that $\mathcal{H}_{\mathcal{E}}$ does contain the embedded image of $\ell^2(G^0, \mu)$ for a certain measure μ, but this space is not typically dense, and almost never equal to $\mathcal{H}_{\mathcal{E}}$. Also, $\mathcal{H}_{\mathcal{E}}$ embeds isometrically into a subspace of $\ell^2(G^1, c)$, but it generally nontrivial to determine whether a given element of $\ell^2(G^1, c)$ lies in this subspace. $\mathcal{H}_{\mathcal{E}}$ may also be understood as a ℓ^2 space of random variables [JP09c, §15.1] or realized as a subspace of $L^2(S', \mathbb{P})$, where S' is a certain space of distributions [JP09a], but both of these constructions are beyond the scope of the present paper.

Lemma 2.3. *If $\varphi : G^0 \to \mathbb{R}$ and M_{φ} denotes the multiplication operator defined by $(M_{\varphi}u)(x) = \varphi(x)u(x)$, then M_{φ} is Hermitian if and only if $M_{\varphi} = k\mathbb{I}$, for some $k \in \mathbb{R}$.*

Proof. Choose any representatives for $u, v \in \mathcal{H}_{\mathcal{E}}$. From the formula (1.12),

$$\langle M_{\varphi}u, v \rangle_{\mathcal{E}} = \frac{1}{2} \sum_{x,y \in G^0} c_{xy}(\varphi(x)u(x)v(x) - \varphi(x)u(x)v(y) - \varphi(y)u(y)v(x) + \varphi(y)u(y)v(y)).$$

By comparison with the corresponding expression, this is equal to $\langle u, M_{\varphi}v \rangle_{\mathcal{E}}$ iff $(\varphi(y) - \varphi(x))u(y)v(x) = (\varphi(y) - \varphi(x))u(x)v(y)$. However, since we are free to vary u and v, it must be the case that φ is constant and hence $\varphi = 0$ in $\mathcal{H}_{\mathcal{E}}$. The converse is trivial. $\qquad\square$

2.1. **The evaluation operators L_x and the reproducing kernel $\{v_x\}$.**

Definition 2.4. For any vertex $x \in G^0$, define the linear evaluation operator L_x on $\mathcal{H}_{\mathcal{E}}$ by

$$L_x u := u(x) - u(o). \tag{2.2}$$

Lemma 2.5. *For any $x \in G^0$, one has $|L_x u| \leq k \mathcal{E}(u)^{1/2}$, where k depends only on x.*

Proof. Since G is connected, choose a path $\{x_i\}_{i=0}^n$ with $x_0 = o$, $x_n = x$ and $c_{x_i, x_{i-1}} > 0$ for $i = 1, \ldots, n$. For $k = \left(\sum_{i=1}^n c_{x_i, x_{i-1}}^{-1} \right)^{1/2}$, the Schwarz inequality yields

$$|L_x u|^2 = |u(x) - u(o)|^2 = \left| \sum_{i=1}^n \sqrt{\frac{c_{x_i, x_{i-1}}}{c_{x_i, x_{i-1}}}} (u(x_i) - u(x_{i-1})) \right|^2 \leq k^2 \mathcal{E}(u). \qquad \square$$

Definition 2.6. Let v_x be defined to be the unique element of $\mathcal{H}_{\mathcal{E}}$ for which

$$\langle v_x, u \rangle_{\mathcal{E}} = u(x) - u(o), \qquad \text{for every } u \in \mathcal{H}_{\mathcal{E}}. \tag{2.3}$$

This is justified by Lemma 2.5 and Riesz's lemma. The family of functions $\{v_x\}_{x \in G^0}$ is called the *energy kernel* because of Corollary 2.7. Note that v_o corresponds to a constant function, since $\langle v_o, u \rangle_{\mathcal{E}} = 0$ for every $u \in \mathcal{H}_{\mathcal{E}}$. Therefore, this term may be ignored or omitted.

Corollary 2.7. $\{v_x\}_{x \in G^0}$ *is a reproducing kernel for $\mathcal{H}_{\mathcal{E}}$. Thus,* $\mathrm{span}\{v_x\}$ *is dense in $\mathcal{H}_{\mathcal{E}}$.*

Proof. Choosing representatives with $v_x(o) = 0$, it is trivial to check that $\langle v_x, v_y \rangle_{\mathcal{E}} = v_x(y) = v_y(x)$ and then apply Aronszajn's Theorem [Aro50]. The energy kernel is clearly total, so density is immediate. $\qquad \square$

There is a rich modern literature dealing with reproducing kernels and their manifold application to both continuous analysis problems (see e.g., [AD06, AL08, AAL08, BV03, Zha09]), and infinite discrete stochastic models. One novel aspect of the present work is the use of "relative" (to the reference vertex o) reproducing kernels.

Remark 2.8. There is an alternative construction of $\mathcal{H}_{\mathcal{E}}$ via techniques of von Neumann and Schoenberg [vN32, Sch38b, Sch38a]. The natural notion of distance on G is the *effective resistance metric R*, which is defined in terms of $\mathcal{E}$. Because this metric is negative semidefinite, von Neumann's method gives an embedding $\Phi : G \to \mathcal{H}$ of the metric space (G, R) into a Hilbert space $\mathcal{H}$ in such a way that $R(x, y) = \|\Phi(x) - \Phi(y)\|_{\mathcal{H}}^2$. In [JP09c, §6], it is shown that $\Phi(x) \mapsto v_x$ is a unitary isomorphism of $\mathcal{H}$ onto $\mathcal{H}_{\mathcal{E}}$.

Remark 2.9 (Probabilistic interpretation of v_x). The energy kernel $\{v_x\}$ is intimately related to effective resistance distance $R(x, y)$. In fact, $R(x, o) = v_x(x) - v_x(o) = \mathcal{E}(v_x)$ and similarly, $R(x, y) = \mathcal{E}(v_x - v_y)$. This is discussed in detail in [JP09b], but we give a brief summary here, to help the reader get a feeling for v_x. For a random walk (RW) on a finite network started at the vertex y, let τ_x be the hitting time of x (i.e., the time at which the random walk first reaches x) and define the function

$$u_x(y) = \mathbb{P}[\tau_x < \tau_o | \text{ RW starts at } y].$$

Here, the RW is governed by transition probabilities $p(x, y) = c_{xy}/c(x)$; cf. Remark 3.2. One can show that $v_x = R(x, o)u_x$ is the representative of v_x with $v_x(o) = 0$. Since the range of u_x is $[0, 1]$, one has $0 \leq v_x(y) - v_x(o) \leq v_x(x) - v_x(o) = R(x, o)$. Many other properties

of v_x are similarly clear from this interpretation. For example, it is easy to compute v_x completely on any tree.

2.2. **The finitely supported functions and the harmonic functions.**

Definition 2.10. For $v \in \mathcal{H}_\mathcal{E}$, one says that v has *finite support* iff there is a finite set $F \subseteq G^0$ for which $v(x) = k \in \mathbb{C}$ for all $x \notin F$. Equivalently, the set of functions of finite support in $\mathcal{H}_\mathcal{E}$ is

$$\mathrm{span}\{\delta_x\} = \{u \in \mathrm{dom}\,\mathcal{E} : u(x) = k \text{ for some } k, \text{ for all but finitely many } x \in G^0\}, \quad (2.4)$$

where δ_x is the Dirac mass at x, i.e., the element of $\mathcal{H}_\mathcal{E}$ containing the characteristic function of the singleton $\{x\}$. It is immediate from (1.14) that $\delta_x \in \mathcal{H}_\mathcal{E}$. Define $\mathcal{F}in$ to be the closure of $\mathrm{span}\{\delta_x\}$ with respect to $\mathcal{E}$.

Definition 2.11. The set of harmonic functions of finite energy is denoted

$$\mathcal{H}arm := \{v \in \mathcal{H}_\mathcal{E} : \Delta v(x) = 0, \text{ for all } x \in G^0\}. \quad (2.5)$$

Note that this is independent of choice of representative for v in virtue of (1.10).

Lemma 2.12. *The Dirac masses* $\{\delta_x\}_{x \in G^0}$ *form a reproducing kernel for* Δ. *That is, for any* $x \in G^0$, *one has* $\langle \delta_x, u \rangle_\mathcal{E} = \Delta u(x)$.

Proof. Compute $\langle \delta_x, u \rangle_\mathcal{E} = \mathcal{E}(\delta_x, u)$ directly from formula (1.12). $\qquad\square$

Remark 2.13. Note that one can take the definition of the Laplacian to be the operator A satisfying $\langle \delta_x, u \rangle_\mathcal{E} = Au(x)$. This point of view is helpful, especially when distinguishing between identities in Hilbert space and pointwise equations. For example, if $h \in \mathcal{H}arm$, then Δh and the constant function $\mathbf{1}$ are identified in $\mathcal{H}_\mathcal{E}$ because $\langle u, \Delta h \rangle_\mathcal{E} = \langle u, \mathbf{1} \rangle_\mathcal{E} = 0$, for any $u \in \mathcal{H}_\mathcal{E}$. However, one should not consider a (pointwise) solution of $\Delta u(x) = 1$ to be a harmonic function.

Lemma 2.14. *For any* $x \in G^0$, $\Delta v_x = \delta_x - \delta_o$.

Proof. Using Lemma 2.12, $\Delta v_x(y) = \langle \delta_y, v_x \rangle_\mathcal{E} = \delta_y(x) - \delta_y(o) = (\delta_x - \delta_o)(y)$. $\qquad\square$

Definition 2.15. A *dipole* is any $v \in \mathcal{H}_\mathcal{E}$ satisfying the pointwise identity $\Delta v = \delta_x - \delta_y$ for some vertices $x, y \in G^0$. It is clear from Lemma 2.14 that the energy kernel consists of dipoles, and that $v_x - v_y$ is always a finite-energy solution to $\Delta v = \delta_x - \delta_y$.

The formula $\langle \delta_x, u \rangle_\mathcal{E} = \Delta u(x)$ of Lemma 2.12 is extremely important. Since $\mathcal{F}in$ is the closure of $\mathrm{span}\{\delta_x\}$, it implies that the finitely supported functions and the harmonic functions are orthogonal. This result is sometimes called the "Royden Decomposition" in honour of the analogous theory established by Royden for Riemann surfaces, when it first appeared in [Yam79, Thm. 4.1]. However, the result is incorrect as stated there and the following corrected form may also be found in [Soa94, §VI] or [LP09, §9.3].

Theorem 2.16. $\mathcal{H}_\mathcal{E} = \mathcal{F}in \oplus \mathcal{H}arm$.

Proof. For all $v \in \mathcal{H}_\mathcal{E}$, Lemma 2.12 gives $\langle \delta_x, v \rangle_\mathcal{E} = \Delta v(x)$. Since $\mathcal{F}in = \mathrm{span}\{\delta_x\}$, this equality shows $v \perp \mathcal{F}in$ whenever v is harmonic. Conversely, if $\langle \delta_x, v \rangle_\mathcal{E} = 0$ for every x, then v must be harmonic. Recall that constants functions are 0 in $\mathcal{H}_\mathcal{E}$. $\qquad\square$

Corollary 2.17. $\mathrm{span}\{\delta_x\}$ *is dense in* $\mathcal{H}_\mathcal{E}$ *iff* $\mathcal{H}arm = 0$.

Remark 2.18. Corollary 2.17 is immediate from Theorem 2.16, but we wish to empha-
size the point, as it is not the usual case elsewhere in the literature and leads to unusual
consequences, e.g., one may have

$$u \neq \sum_{x \in G^0} u(x)\delta_x, \quad \text{in } \mathcal{H}_{\mathcal{E}}.$$

More precisely, $\|u - \sum_{x \in G_k} u(x)\delta_x\|_{\mathcal{E}}$ may not tend to 0 as $k \to \infty$, for any exhaustion $\{G_k\}$.
Part of the importance of the energy kernel $\{v_x\}$ arises from the fact that there is no other
natural representing set for $\mathcal{H}_{\mathcal{E}}$. See also [Yam79, Lemma 3.1] for a similar interesting
implication.

Definition 2.19. Let $f_x = P_{\mathcal{F}in}v_x$ denote the image of v_x under the projection to $\mathcal{F}in$. Simi-
larly, let $h_x = P_{\mathcal{H}arm}v_x$ denote the image of v_x under the projection to $\mathcal{H}arm$.

For future reference, we state the following immediate consequence of orthogonality.

Lemma 2.20. *With $f_x = P_{\mathcal{F}in}v_x$, $\{f_x\}_{x \in G^0}$ is a reproducing kernel for $\mathcal{F}in$, but $f_x \perp \mathcal{H}arm$.*
Similarly, with $h_x = P_{\mathcal{H}arm}v_x$, $\{h_x\}_{x \in G^0}$ is a reproducing kernel for $\mathcal{H}arm$, but $h_x \perp \mathcal{F}in$.

While it may not be true that v_y is in span$\{\delta_x\}$ (or even in its closure), the following
result shows that δ_y is always in span$\{v_x\}$ when deg$(y) < \infty$.

Lemma 2.21. *For any $x \in G^0$, $\delta_x = c(x)v_x - \sum_{y \sim x} c_{xy}v_y$.*

Proof. Lemma 2.12 implies $\langle \delta_x, u \rangle_{\mathcal{E}} = \langle c(x)v_x - \sum_{y \sim x} c_{xy}v_y, u \rangle_{\mathcal{E}}$ for every $u \in \mathcal{H}_{\mathcal{E}}$, so apply
this to $u = v_z$, $z \in G^0$. Since $\delta_x, v_x \in \mathcal{H}_{\mathcal{E}}$, it must also be that $\sum_{y \sim x} c_{xy}v_y \in \mathcal{H}_{\mathcal{E}}$. □

Remark 2.22 (Real and complex-valued functions on G^0). While we will need complex-
valued functions for some results obtained via spectral theory, it will usually suffice to
consider $\mathbb{R}$-valued functions because of the following lemma.

Lemma 2.23. *The reproducing kernels v_x, f_x, h_x are all $\mathbb{R}$-valued functions.*

Proof. Computing directly,

$$\langle \overline{v_z}, \overline{u} \rangle_{\mathcal{E}} = \frac{1}{2} \sum_{x,y \in G^0} (v_z(x) - v_z(y))(\overline{u}(x) - \overline{u}(y)) = \overline{\langle v_z, u \rangle_{\mathcal{E}}}.$$

Then applying the reproducing kernel property,

$$\overline{\langle v_z, u \rangle_{\mathcal{E}}} = \overline{u(x) - u(o)} = \overline{u}(x) - \overline{u}(o) = \langle v_z, \overline{u} \rangle_{\mathcal{E}}.$$

Thus $\langle \overline{v_z}, \overline{u} \rangle_{\mathcal{E}} = \langle v_z, \overline{u} \rangle_{\mathcal{E}}$ for every $u \in \mathcal{H}arm$, and v_z must be $\mathbb{R}$-valued. The same compu-
tation applies to f_z and h_z. □

Definition 2.24. A sequence $\{u_n\} \subseteq \mathcal{H}_{\mathcal{E}}$ *converges pointwise in $\mathcal{H}_{\mathcal{E}}$* iff $\exists k \in \mathbb{C}$ such that
$u_n(x) - u(x) \to k$, for all $x \in G^0$.

Lemma 2.25. *If $\{u_n\}$ converges to u in $\mathcal{E}$, then $\{u_n\}$ converges to u pointwise in $\mathcal{H}_{\mathcal{E}}$.*

Proof. Define $w_n := u_n - u$ so that $\|w_n\|_{\mathcal{E}} \to 0$. Then

$$|w_n(x) - w_n(o)| = |\langle v_x, w_n \rangle_{\mathcal{E}}| \leq \|v_x\|_{\mathcal{E}} \cdot \|w_n\|_{\mathcal{E}} \xrightarrow{n \to \infty} 0,$$

so that $\lim w_n$ exists pointwise and is a constant function. □

3. The discrete Gauss-Green formula

In Theorem 3.12, we establish a discrete version of the Gauss-Green formula which extends Proposition 1.9 to the case of infinite graphs; the scope of validity of this formula is given in terms of the space $\mathcal{M}$ of Definition 3.1. We are able to prove in Lemma 5.8 that the boundary term vanishes for elements of span$\{v_x\}$ and in Lemma 5.15 that it vanishes for finitely supported functions. Corollary 5.6 recovers the well-known fact that nontrivial harmonic functions cannot be in $\ell^2(G^0)$; see also the beginning of §4.

A key difference between our development of the relationship between the Laplace operator Δ and the Dirichlet energy form $\mathcal{E}$ is that Δ is Hermitian but not necessarily self-adjoint in the present context. This is in sharp contrast to the literature on Dirichlet spaces in potential theory [Bre67, CC72] and the general theory of Dirichlet forms and probability [FŌT94, BH91]. In fact, the "gap" between Δ and its self-adjoint extensions comprises an important part of the boundary theory for (G, c), and accounts for features of the boundary terms in the discrete Gauss-Green identity of Theorem 3.12.

3.1. Monopoles and the domain of Δ.

Definition 3.1. A *monopole* at $x \in G^0$ is an element $w_x \in \mathcal{H}_\mathcal{E}$ which satisfies $\Delta w_x(y) = \delta_{xy}$, where $k \in \mathbb{C}$ and δ_{xy} is Kronecker's delta. When nonempty, the set of monopoles at the origin is closed and convex, so $\mathcal{E}$ attains a unique minimum here; let w_o always denote the unique energy-minimizing monopole at the origin.

When $\mathcal{H}_\mathcal{E}$ contains monopoles, let $\mathcal{M}_x$ denote the vector space spanned by the monopoles at x. This implies that $\mathcal{M}_x$ may contain harmonic functions; see Lemma 4.1. We indicate the distinguished monopoles

$$w_x^v := v_x + w_o \quad \text{and} \quad w_x^f := f_x + w_o, \tag{3.1}$$

where $f_x = P_{\mathcal{F}in}v_x$. (Corollary 4.4 below confirms that $w_x^v = w_x^f$ for all x iff if $\mathcal{H}arm = 0$.)

Remark 3.2 (Monopoles and transience). The presence of monopoles in $\mathcal{H}_\mathcal{E}$ is equivalent to the transience of the underlying network, that is, the transience of the simple random walk on the network with transition probabilities $p(x, y) = c_{xy}/c(x)$. To see this, note that if w_x is a monopole, then the current induced by w_x is a unit current flow to infinity with finite energy. It was proved in [Yam79] that the network supports monopoles if and only if the Green kernel $\mathcal{G}$ exists (i.e., if and only if the random walk on the network is transient). This also appears in [Lyo83] in the form that the network is transient if and only if there exists a unit current flow to infinity; see [LP09, Thm. 2.10].

Let $p_n(x, y)$ denote the probability that the random walk started at x is at y after n steps. If $\gamma = (x = x_0, x_1, \ldots, x_n = y)$ is a path from x to y, then define

$$\mathbb{P}(\gamma) = \prod_{i=1}^{n} \frac{c_{x_{i-1}x_i}}{c(x)}.$$

Let $\Gamma(x, y)$ be the set of all finite paths from x to y, and let $\Gamma_n(x, y)$ be the subset of paths of length n. Then $p_1(x, y) = p(x, y)$ as defined just above, and in general,

$$p_n(x, y) = \sum_{\gamma \in \Gamma_n(x,y)} \mathbb{P}(\gamma).$$

Suppose the Green kernel $\mathcal{G}$ is defined in the usual manner:

$$\mathcal{G}(x, y) = \sum_{n=0}^{\infty} p_n(x, y). \qquad (3.2)$$

Now let us denote the *symmetrized Green kernel* by $g_*(x, y) = \mathcal{G}(x, y)/c(y)$, as in [Kig09a, §2]. Then $g_*(x, y) = g_*(y, x)$ and there is a simple formula for g_* in terms of the (wired) resistance metric on G; see [Kig09a, (2.2)] and [JP09b]. For the moment, let us abuse notation and use w_x^f to denote the representative of w_x^f which vanishes at ∞. Then the relationship between the monopoles and the Green kernel is given by

$$w_x^f(y) = g_*(x, y) = \frac{\mathcal{G}(x, y)}{c(y)}. \qquad (3.3)$$

Roughly speaking, the Green kernel is a reproducing kernel for the subspace of dom $\mathcal{E}$ which consists of $\mathcal{E}$-limits of finitely supported functions. The monopole domain $\mathcal{M}$ contains an extension of the Green kernel which is a reproducing kernel for all of dom $\mathcal{E}$.

It is well-known that the transience/recurrence of the random walk is independent of the vertex at which the walk is started; equivalently, $\mathcal{G}(x, y)$ is finite for some x and y if and only if it is finite for all x, y. The corresponding statement for monopoles is also clear from (3.1): there exists a monopole at some vertex o if and only if there exists a monopole at any other vertex x.

Remark 3.3. Note that $w_o \in \mathcal{F}in$, whenever it is present in $\mathcal{H}_\mathcal{E}$, and similarly that w_x^f is the energy-minimizing element of $\mathcal{M}_x$. To see this, suppose w_x is any monopole at x. Since $w_x \in \mathcal{H}_\mathcal{E}$, write $w_x = f + h$ by Theorem 2.16, and get $\mathcal{E}(w_x) = \mathcal{E}(f) + \mathcal{E}(h)$. Projecting away the harmonic component will not affect the monopole property, so $w_x^f = P_{\mathcal{F}in}w_x$ is the unique monopole of minimal energy. Also, w_o corresponds to the projection of $\mathbf{1}$ to $\mathbf{D}_0$; see §4.1.

Definition 3.4. The dense subspace of $\mathcal{H}_\mathcal{E}$ spanned by monopoles (or dipoles) is

$$\mathcal{M} := \mathrm{span}\{v_x\}_{x \in G^0} + \mathrm{span}\{w_x^v, w_x^f\}_{x \in G^0}. \qquad (3.4)$$

Let $\Delta_\mathcal{M}$ be the closure of the Laplacian when taken to have the dense domain $\mathcal{M}$.

Note that $\mathcal{M} = \mathrm{span}\{v_x\}$ when there are no monopoles (i.e., when all solutions of of $\Delta w = \delta_x$ have infinite energy), and that $\mathcal{M} = \mathrm{span}\{w_x^v, w_x^f\}$ when there are monopoles; see Lemma 4.1.

The space $\mathcal{M}$ is introduced as a dense domain for Δ (see Remark 3.8) and for its use as a hypothesis in our main result, that is, as the largest domain of validity for the discrete Gauss-Green identity of Theorem 3.12. Note that while a general monopole need not be in dom $\Delta_\mathcal{M}$ (see [JP09c, Ex. 13.8 or Ex. 14.39]), we show in Lemma 3.7 that it is always the case that it lies in dom $\Delta_\mathcal{M}^*$.

Definition 3.5. A Hermitian operator S on a Hilbert space $\mathcal{H}$ is called *semibounded* iff

$$\langle v, Sv \rangle \geq 0, \qquad \text{for every } v \in \mathcal{D}, \qquad (3.5)$$

so that its spectrum lies in some halfline $[\kappa, \infty)$ and its defect indices agree.

Lemma 3.6. $\Delta_\mathcal{M}$ *is Hermitian; a fortiori,* $\Delta_\mathcal{M}$ *is semibounded.*

Proof. Suppose we have two finite sums $u = \sum a_x w_x$ and $v = \sum b_y w_y$, writing w_x for w_x^v or w_x^f. We may assume that o appears neither in the sum u nor for v; see Definition 2.6. Then Lemma 2.12 gives

$$\langle u, \Delta v \rangle_{\mathcal{E}} = \sum \overline{a_x} b_y \langle w_x, \Delta w_y \rangle_{\mathcal{E}} = \sum \overline{a_x} b_y \langle w_x, \delta_y \rangle_{\mathcal{E}} = \sum \overline{a_x} b_y \Delta w_x(y) = \sum \overline{a_x} b_y \delta_{xy}.$$

Of course, $\langle \Delta u, v \rangle_{\mathcal{E}} = \sum \overline{a_x} b_y \delta_{xy}$ exactly the same way. The argument for linear combinations from $\{v_x\}$ is similar, so $\Delta_{\mathcal{M}}$ is Hermitian. Then

$$\langle u, \Delta u \rangle_{\mathcal{E}} = \sum_{x,y} \overline{a_x} a_y \delta_{xy} = \sum_x |a_x|^2 \geq 0$$

shows $\Delta_{\mathcal{M}}$ is semibounded. The argument for $\{v_x\}$ is similar. $\qquad\square$

Since Δ agrees with $\Delta_{\mathcal{M}}$ pointwise, we may suppress reference to the domain for ease of notation. When given a pointwise identity $\Delta u = v$, there is an associated identity in $\mathcal{H}_{\mathcal{E}}$, but the Lemma 3.7 shows that one must use the adjoint.

Lemma 3.7. *For $u, v \in \mathcal{H}_{\mathcal{E}}$, $\Delta u = v$ pointwise if and only if $v = \Delta_{\mathcal{M}}^* u$ in $\mathcal{H}_{\mathcal{E}}$.*

Proof. We show that $u \in \operatorname{dom} \Delta_{\mathcal{M}}^*$ for simplicity, so let $\varphi \in \operatorname{span}\{v_x\}$ be given by $\varphi = \sum_{i=1}^n a_i v_{x_i}$; the proof for $\varphi \in \operatorname{span}\{w_x^v, w_x^f\}$ is similar. Then Lemma 2.12 and Lemma 2.14 give

$$\langle \Delta \varphi, u \rangle_{\mathcal{E}} = \sum_{i=1}^n a_i \langle \delta_{x_i} - \delta_o, u \rangle_{\mathcal{E}} = \sum_{i=1}^n a_i (\Delta u(x_i) - \Delta u(o)).$$

Since $\Delta u(x) = v(x)$ by hypothesis, this may be continued as

$$\langle \Delta \varphi, u \rangle_{\mathcal{E}} = \sum_{i=1}^n a_i (v(x_i) - v(o)) = \sum_{i=1}^n a_i \langle v_{x_i}, v \rangle_{\mathcal{E}} = \langle \varphi, v \rangle_{\mathcal{E}}.$$

Then the Schwarz inequality gives the estimate $|\langle \Delta \varphi, u \rangle_{\mathcal{E}}| = |\langle \varphi, v \rangle_{\mathcal{E}}| \leq \|\varphi\|_{\mathcal{E}} \|v\|_{\mathcal{E}}$, which means $u \in \operatorname{dom} \Delta_{\mathcal{M}}^*$. The converse is trivial. $\qquad\square$

Remark 3.8 (Monopoles give a reproducing kernel for ran $\Delta_{\mathcal{M}}$). Lemma 3.7 means that

$$\langle w_x, \Delta u \rangle_{\mathcal{E}} = \langle \delta_x, u \rangle_{\mathcal{E}}, \qquad \text{for all } u \in \operatorname{dom} \Delta_{\mathcal{M}}. \tag{3.6}$$

for every $w_x \in \mathcal{M}_x$. Combined with Lemma 2.12, this immediately gives

$$\langle w_x, \Delta u \rangle_{\mathcal{E}} = \Delta u(x). \tag{3.7}$$

If $\{w_x\}_{x \in G^0}$ is a collection of monopoles which includes one element from each $\mathcal{M}_x$, then this collection is a reproducing kernel for ran $\Delta_{\mathcal{M}}$. Note that the expression $\Delta u(x)$ is defined in terms of differences, so the right-hand side is well-defined even without reference to another vertex, i.e., independent of any choice of representative.

In Definition 3.4, we give a domain $\mathcal{M}$ for Δ which ensures that ran $\Delta_{\mathcal{M}}$ contains all finitely supported functions and is thus dense in $\mathcal{F}in$. However, even when Δ is defined so as to be a closed operator, one may not have $\mathcal{F}in \subseteq \operatorname{ran} \Delta$; in general, the containment $\operatorname{ran}(S^{\operatorname{clo}}) \subseteq (\operatorname{ran} S)^{\operatorname{clo}}$ may be strict. The operator closure S^{clo} is done with respect to the graph norm, and the closure of the range is done with respect to $\mathcal{E}$. We note that [MYY94, (G.1)] claims that the Green function is a reproducing kernel for all of $\mathcal{F}in$. In our context,

at least, the Green function is a reproducing kernel only for $\operatorname{ran}\Delta$, where Δ has been chosen with a suitable dense domain. In general, the containment $\operatorname{ran}\Delta \subseteq \mathcal{F}in$ may be strict. In fact, it is true that $\operatorname{ran}\Delta_{\mathcal{M}}^{*} \subseteq \mathcal{F}in$, and even this containment may be strict. Note that w_x^f is the only element of $\mathcal{M}_x$ which lies in $(\operatorname{ran}\Delta_{\mathcal{M}})^{\text{clo}}$, and it may not lie in $\operatorname{ran}\Delta_{\mathcal{M}}$.

A different choice of domain for Δ can exacerbate the discrepancy between $\operatorname{ran}\Delta$ and $\mathcal{F}in$: if one were to define $\Delta_{\mathcal{V}}$ to be the closure of Δ when taken to have dense domain $\mathcal{V} := \operatorname{span}\{v_x\}$, then $\operatorname{ran}\Delta_{\mathcal{V}}$ is dense in $\mathcal{F}in_2$, the $\mathcal{E}$-closure of $\operatorname{span}\{\delta_x - \delta_o\}$. However, it can happen that $\mathcal{F}in_2$ is a proper orthogonal subspace of $\mathcal{F}in$ (the $\mathcal{E}$-closure of $\operatorname{span}\{\delta_x\}$). This is discussed further in [JP09c, §8]; an example of $f \in \mathcal{F}in_1 := \mathcal{F}in \ominus \mathcal{F}in_2$ is computed in [JP09c, Ex. 14.35]. The domain of Δ can thus induce a refinement of the Royden decomposition:

$$\mathcal{H}_{\mathcal{E}} = \mathcal{F}in_1 \oplus \mathcal{F}in_2 \oplus \mathcal{H}arm.$$

See Theorem 2.16 and the comment preceding it. This highlights the importance of the choice of $\mathcal{M}$ as the domain for Δ.

3.2. **Relating Δ to $\mathcal{E}$.**

Definition 3.9. If H is a subgraph of G, then the boundary of H is

$$\operatorname{bd}H := \{x \in H : \exists y \in H^{\complement}, y \sim x\}. \tag{3.8}$$

The *interior* of a subgraph H consists of the vertices in H whose neighbours also lie in H:

$$\operatorname{int}H := \{x \in H : y \sim x \implies y \in H\} = H \setminus \operatorname{bd}H. \tag{3.9}$$

For vertices in the boundary of a subgraph, the *normal derivative* of v is

$$\frac{\partial v}{\partial \mathrm{n}}(x) := \sum_{y \in H} c_{xy}(v(x) - v(y)), \qquad \text{for } x \in \operatorname{bd}H. \tag{3.10}$$

Thus, the normal derivative of v is computed like $\Delta v(x)$, except that the sum extends only over the neighbours of x which lie in H.

Definition 3.9 will be used primarily for subgraphs that form an exhaustion of G, in the sense of Definition 1.5: an increasing sequence of finite and connected subgraphs $\{G_k\}$, so that $G_k \subseteq G_{k+1}$ and $G = \bigcup G_k$. Also, recall that $\sum_{\operatorname{bd}G} := \lim_{k\to\infty}\sum_{\operatorname{bd}G_k}$ from Definition 3.10.

Definition 3.10. A *boundary sum* is computed in terms of an exhaustion $\{G_k\}$ by

$$\sum_{\operatorname{bd}G} := \lim_{k\to\infty}\sum_{\operatorname{bd}G_k}, \tag{3.11}$$

whenever the limit is independent of the choice of exhaustion, as in Definition 1.6.

Remark 3.11 (bd G as a measure space). The boundary $\operatorname{bd}G$ is formalized as a measure space in [JP09a, JP09d]; see also [JP09c, §7]. Just as the Martin boundary is a support set for a representing measure for the nonnegative harmonic functions, and the Poisson boundary is a support set for a representing measure for the bounded harmonic functions, it is shown in [JP09a] that $\operatorname{bd}G$ is a support set for a representing measure for the harmonic functions of finite energy. For more about Martin and Poisson boundaries, see [Saw97, Woe00] and the references therein.

1   **Theorem 3.12** (Discrete Gauss-Green Formula)**.** *If $u \in \mathcal{H}_{\mathcal{E}}$ and $v \in \mathcal{M}$, then*

$$\langle u, v \rangle_{\mathcal{E}} = \sum_{G^0} \overline{u} \Delta v + \sum_{\mathrm{bd}\, G} \overline{u} \tfrac{\partial v}{\partial \mathrm{n}}. \qquad (3.12)$$

2   *Proof.* It suffices to work with $\mathbb{R}$-valued functions and then complexify afterwards. By the
3   same computation as in Proposition 1.9, we have

$$\frac{1}{2} \sum_{x,y \in G_k} c_{xy}(\overline{u}(x) - \overline{u}(y))(v(x) - v(y)) = \sum_{x \in \mathrm{int}\, G_k} \overline{u}(x) \Delta v(x) + \sum_{x \in \mathrm{bd}\, G_k} \overline{u}(x) \tfrac{\partial v}{\partial \mathrm{n}}(x). \qquad (3.13)$$

4   Taking limits of both sides as $k \to \infty$ gives (3.12). It remains to see that one of the sums
5   on the right-hand side is finite (and hence that both are). Note that if $v = w_z$ is a monopole,
6   then

$$\sum_{x \in G^0} u(x) \Delta v(x) = \sum_{x \in G^0} u(x) \delta_z(x) = u(z).$$

7   This is obviously independent of exhaustion, and immediately extends to $v \in \mathcal{M}$.   □

8   The following boundary representation of harmonic functions is the focus of [JP09a].

9   **Corollary 3.13** (Boundary representation of harmonic functions)**.** *For all $u \in \mathcal{H}arm$,*

$$u(x) = \sum_{\mathrm{bd}\, G} u \tfrac{\partial h_x}{\partial \mathrm{n}} + u(o). \qquad (3.14)$$

10   *Proof.* Lemma 2.23 and (2.3) imply $u(x) - u(o) = \langle v_x, u \rangle_{\mathcal{E}} = \overline{\langle u, v_x \rangle_{\mathcal{E}}} = \sum_{\mathrm{bd}\, G} u \tfrac{\partial h_x}{\partial \mathrm{n}}$.   □

11   **Lemma 3.14.** *For all $u \in \mathrm{dom}\, \Delta_{\mathcal{M}}$, $\sum_{G^0} \Delta u = - \sum_{\mathrm{bd}\, G} \tfrac{\partial u}{\partial \mathrm{n}}$. Thus, the discrete Gauss-Green*
12   *formula (3.12) is independent of choice of representatives.*

13   *Proof.* On each G_k, each edge appears twice in the sum (once with each sign/orientation)
14   and so

$$\sum_{x \in \mathrm{int}\, G_k} \Delta u(x) + \sum_{x \in \mathrm{bd}\, G_k} \tfrac{\partial u}{\partial \mathrm{n}}(x) = \sum_{x,y \in G_k} c_{xy}(u(x) - u(y)) = 0.$$

15   To check a different representative, use the first part to compute

$$\sum_{G^0}(u+k)\Delta v + \sum_{\mathrm{bd}\, G}(u+k)\tfrac{\partial v}{\partial \mathrm{n}} = \sum_{G^0} u \Delta v + \sum_{\mathrm{bd}\, G} u \tfrac{\partial v}{\partial \mathrm{n}} + k\left(\cancel{\sum_{G^0} \Delta v} + \cancel{\sum_{\mathrm{bd}\, G} \tfrac{\partial v}{\partial \mathrm{n}}} \right). \qquad □$$

16   *Remark* 3.15. It is clear that (3.12) remains true much more generally than under the
17   specified conditions; certainly the formula holds whenever $\sum_{x \in G^0} |u(x)\Delta v(x)| < \infty$. Un-
18   fortunately, given any hypotheses more specific than this, the limitless variety of infinite
19   networks almost always allow one to construct a counterexample; i.e. one cannot give a
20   condition for which the formula is true for all $u \in \mathcal{H}_{\mathcal{E}}$, for all networks. To see this, suppose
21   that $v = \sum_{i=1}^{\infty} a_i w_{x_i}$ with each w_{x_i} a monopole at the vertex x_i. Then

$$\sum_{x \in G^0} u(x) \Delta v(x) = \sum_{i=1}^{\infty} a_i u(x_i),$$

and one would need to provide a condition on sequences $\{a_i\}$ that would ensure $\sum_{i=1}^{\infty} a_i u(x_i)$ is absolutely convergent for all $u \in \mathcal{H}_{\mathcal{E}}$. Such a hypothesis is not likely to be useful (if it is even possible to construct) and would depend heavily on the network under investigation. Nonetheless, the formula remains true and even useful in many specific and general contexts. For example, it is clearly valid whenever v is a dipole, including all those in the energy kernel. We will also see that it holds for the projections of v_x to $\mathcal{F}in$ and to $\mathcal{H}arm$. Consequently, for v which are limits of elements in $\mathcal{M}$, we can use this result in combination with ad hoc arguments.

Remark 3.16. We refer to $\sum_{\mathrm{bd}\,G} u \frac{\partial v}{\partial \mathrm{n}}$ as the "boundary term" by analogy with classical PDE theory. This terminology should not be confused with the notion of boundary that arises in the discussion of the discrete Dirichlet problem, where the boundary is a prescribed set of G^0. As the boundary term may be difficult to contend with, it is extremely useful to know when it vanishes, for example:

  (i) when the network is recurrent (Lemma 4.6),
  (ii) when v is an element of the energy kernel (Lemma 5.8),
  (iii) when $u, v, \Delta u, \Delta v$ lie in ℓ^2 (Lemma 5.12), and
  (iv) when either u or v has finite support (Lemma 5.15).

In fact, Lemma 4.6 expresses the fact that it is precisely the presence of monopoles that prevents the boundary term from vanishing.

4. More about monopoles and the space $\mathcal{M}$

This section studies the role of the monopoles with regard to the boundary term of Theorem 3.12, and provides several characterizations of transience of the network, in terms the operator-theoretic properties of $\Delta_{\mathcal{M}}$.

Note that if $h \in \mathcal{H}arm$ satisfies the hypotheses of Theorem 3.12, then $\mathcal{E}(h) = \sum_{\mathrm{bd}\,G} h \frac{\partial h}{\partial \mathrm{n}}$. In Theorem 4.6 we show that $\mathcal{E}(u) = \sum_{G^0} u \Delta u$ for all $u \in \mathcal{H}_{\mathcal{E}}$ iff the network is recurrent. With respect to $\mathcal{H}_{\mathcal{E}} = \mathcal{F}in \oplus \mathcal{H}arm$, this shows that the energy of finitely supported functions comes from the sum over G^0, and the energy of harmonic functions comes from the boundary sum. However, for a monopole w_x, the representative specified by $w_x(x) = 0$ satisfies $\mathcal{E}(w) = \sum_{\mathrm{bd}\,G} w \frac{\partial w}{\partial \mathrm{n}}$ but the representative specified by $w_x(x) = \mathcal{E}(w_x)$ satisfies $\mathcal{E}(w) = \sum_{G^0} w \Delta w$. Roughly speaking, a monopole is therefore "half of a harmonic function" or halfway to being a harmonic function. A further justification for this comment is given by Corollary 4.4: the proof shows that a harmonic function can be constructed from two monopoles at the same vertex. A different perspective one the same theme is given in Remark 4.14. The general theme of this section is the ability of monopoles to "bridge" the finite and the harmonic.

Lemma 4.1. *When the network is transient, $\mathcal{M}$ contains the spaces* $\mathrm{span}\{v_x\}$, $\mathrm{span}\{f_x\}$, *and* $\mathrm{span}\{h_x\}$, *where* $f_x = P_{\mathcal{F}in}v_x$ *and* $h_x = P_{\mathcal{H}arm}v_x$.

Proof. The first two are obvious, since $v_x = w_x^v - w_o$ and $f_x = w_x^f - w_o$ by Definition 3.1. For the harmonics, note that these same identities give

$$w_x^v - w_o = v_x = f_x + h_x = w_x^f - w_o + h_x,$$

which implies that $h_x = w_x^v - w_x^f$. (Of course, $w_x^v = w_x^f$ when $\mathcal{H}arm = 0$.) □

Theorem 4.2 ([Soa94], Thm. 1.33). *Let u be a nonnegative function on a recurrent network. Then u is superharmonic if and only if u is constant.*

Corollary 4.3. *If $\mathcal{H}arm \neq 0$, then there is a monopole in $\mathcal{H}_{\mathcal{E}}$.*

Proof. If $h \in \mathcal{H}arm$ and $h \neq 0$, then $h = h_1 - h_2$ with $h_i \in \mathcal{H}arm$ and $h_i \geq 0$, by [Soa94, Thm. 3.72]. (Here, $h_i \geq 0$ means that h_i is bounded below, and so we can choose a representative which is nonnegative.) Since the h_i cannot both be 0, Theorem 4.2 implies the network is transient. Then by [Lyo83, Thm. 1], the network supports a monopole. □

Corollary 4.4. $\mathcal{H}arm \neq 0$ *iff there is more than one monopole at one (i.e. every) vertex x.*

Proof. If $\mathcal{H}_{\mathcal{E}}$ contains a monopole $w_x \neq w_x^v$, then $h := w_x^v - w_x$ is a nonzero element of $\mathcal{H}arm$. Conversely, if $\mathcal{H}arm \neq 0$, then there is a monopole $w \in \mathcal{H}_{\mathcal{E}}$ by Corollary 4.3. For a nonzero element $h \in \mathcal{H}arm$, $w + h$ is also a monopole. □

Definition 4.5. The phrase "*the boundary term is nonvanishing*" indicates that (3.12) holds with nonzero boundary sum when applied to $\langle u, v \rangle_{\mathcal{E}}$, for every representative of u except one; namely, the one specified by $u(x) = \langle u, w_x^v \rangle_{\mathcal{E}}$.

Recall from Remark 3.2 that the network is transient iff there are monopoles in $\mathcal{H}_{\mathcal{E}}$.

Theorem 4.6. *The network is transient if and only if the boundary term is nonvanishing.*

Proof. ($\Rightarrow$) If the network is transient, then as explained in Remark 3.2, there is a $w \in \mathcal{H}_{\mathcal{E}}$ with $\Delta w = \delta_z$. Now let $w_z := P_{\mathcal{F}in}w$ so that for any $u \in \mathrm{dom}\,\Delta_{\mathcal{M}}$, (3.12)

$$\langle u, w_z \rangle_{\mathcal{E}} = u(z) + \sum_{\mathrm{bd}\,G} u \frac{\partial w_z}{\partial \mathrm{n}}.$$

It is immediate that $\sum_{\mathrm{bd}\,G} u \frac{\partial w_z}{\partial \mathrm{n}} = 0$ if and only if the computation is done with the representative of u specified by $u(z) = \langle u, w_z \rangle_{\mathcal{E}}$.

($\Leftarrow$) Suppose that there does not exist $w \in \mathcal{H}_{\mathcal{E}}$ with $\Delta w = \delta_z$, for any $z \in G^0$. Then $\mathcal{M} = \mathrm{span}\{v_x\}$ as discussed in Definition 3.1. Therefore, it suffices to show that

$$\langle u, v_x \rangle_{\mathcal{E}} = \sum_{x \in G^0} u \Delta v_x,$$

but this is clear because both sides are equal to $u(x) - u(o)$ by (2.3) and Lemma 2.14. □

The next theorem shows that monopoles may be constructed explicitly as weak-$*$ limits in the Hilbert space $\mathcal{H}_{\mathcal{E}}$. Nonuniqueness of weak-$*$ limits may lead to nonuniqueness of monopoles at a given point x (provided $\mathcal{H}arm \neq 0$; see Corollary 4.4). One would like to argue as follows (as suggested by a referee): for $\varepsilon > 0$, let D_ε be the multiplication operator corresponding to diagonal matrix with entries $\frac{c(x)}{\varepsilon + c(x)} < 1$. Then with $p(x, y) = c_{xy}/c(x)$ giving transition probabilities as before, and defining the heat operator pointwise by $Pu(x) = \sum_{y \in G^0} p(x, y)u(y)$, one can see that the row sums of the corresponding matrix satisfy

$$(D_\varepsilon P \mathbf{1})(x) = \sum_{y \in G^0} \frac{c(x)}{\varepsilon + c(x)} \frac{c_{xy}}{c(x)} = \frac{c(x)}{\varepsilon + c(x)} < 1, \qquad \text{for each fixed } x \in G^0.$$

Consequently,

$$(\varepsilon + \Delta)^{-1} = \frac{1}{\varepsilon + c}(\mathbb{I} - D_\varepsilon P)^{-1} = \frac{1}{\varepsilon + c} \sum_{n=0}^{\infty} (D_\varepsilon P)^n \xrightarrow{\varepsilon \to 0} \frac{1}{c} \sum_{n=0}^{\infty} p(x, y)^n. \qquad (4.1)$$

It is well known that the network is transient if and only if the entries of the matrix $\sum_{n=0}^{\infty} p(x, y)^n$ corresponding to the Green operator are finite. However, this runs into difficulties in the case when c is unbounded, and so we make use of the spectral theorem.

Theorem 4.7. *The network is transient if and only if $f_k := (\varepsilon_k + \Delta_{\mathcal{M}})^{-1}\delta_x$ is weak-$*$ convergent for some sequence $\varepsilon_k \to 0$.*

Proof. We show that there is a monopole if and only if there is sequence $\{\varepsilon_k\}$ with $\varepsilon_k \to 0$ and $\sup_k \|(\varepsilon_k + \Delta_{\mathcal{M}})^{-1}\delta_x\|_{\mathcal{E}} \leq B < \infty$.

($\Rightarrow$) Let $\Delta_{\mathcal{M}}$ be any self-adjoint extension of $\Delta_{\mathcal{M}}$ whose spectrum lies in $[0, \infty)$, and let $E(d\lambda)$ be the corresponding projection-valued measure. For concreteness, one may take the Friedrichs extension, but this is not necessary; Δ commutes with conjugation, and so a theorem of von Neumann implies that such an extension exists. Then

$$R_{\varepsilon}u = (\varepsilon + \Delta_{\mathcal{M}})^{-1}u = \int_0^{\infty} \frac{1}{\varepsilon + \lambda}E(d\lambda)u, \qquad (4.2)$$

where we use the notation $R_{\varepsilon} := (\varepsilon + \Delta_{\mathcal{M}})^{-1}$ for the resolvent. Note that $\Delta_{\mathcal{M}}R_{\varepsilon} \subseteq (\Delta_{\mathcal{M}}R_{\varepsilon})^* = R_{\varepsilon}^*\Delta_{\mathcal{M}}^* = R_{\varepsilon}\Delta_{\mathcal{M}}$. On the other hand, $\Delta_{\mathcal{M}} \subseteq \Delta_{\mathcal{M}}^*$ and therefore $R_{\varepsilon}\Delta_{\mathcal{M}} \subseteq R_{\varepsilon}\Delta_{\mathcal{M}}^*$. Combining these gives $\Delta_{\mathcal{M}}R_{\varepsilon} \subseteq R_{\varepsilon}\Delta_{\mathcal{M}}^*$. Now we apply this and (4.2) to $u = \Delta^*w$ to get

$$f_k = (\varepsilon_k + \Delta_{\mathcal{M}})^{-1}\delta_x = (\varepsilon_k + \Delta_{\mathcal{M}})^{-1}\Delta_{\mathcal{M}}^*w = \Delta_{\mathcal{M}}(\varepsilon_k + \Delta_{\mathcal{M}})^{-1}w = \int_0^{\infty} \frac{\lambda}{\varepsilon_k + \lambda}E(d\lambda)w.$$

Note that R_{ε} is bounded, and so $w \in \text{dom}\, R_{\varepsilon}$ automatically. This integral implies

$$\|f_k\|_{\mathcal{E}}^2 \leq \int_0^{\infty} \left(\frac{\lambda}{\varepsilon_k + \lambda}\right)^2 \|E(d\lambda)w\|_{\mathcal{E}}^2 \leq \int_0^{\infty} \|E(d\lambda)w\|_{\mathcal{E}}^2 = \|w\|_{\mathcal{E}}^2.$$

Thus we have $\sup_k \|(\varepsilon_k + \Delta)^{-1}\delta_x\|_{\mathcal{E}} = \sup \|f_k\|_{\mathcal{E}} \leq B = \|w\|_{\mathcal{E}} < \infty$.

($\Leftarrow$) We show the existence of a monopole at x. Since $\varepsilon_k f_k + \Delta f_k = \delta_x$, the bound $\sup \|f_k\|_{\mathcal{E}} \leq B$ implies that

$$\|\Delta f_k - \delta_x\|_{\mathcal{E}} = \|\varepsilon_k f_k\| \leq \varepsilon_k B \to 0.$$

Let w be a weak-$*$ limit of $\{f_k\}$. Then for $\varphi \in \text{dom}\,\Delta_{\mathcal{M}}$,

$$\langle \Delta\varphi, w\rangle_{\mathcal{E}} = \lim_{k\to\infty}\langle \Delta\varphi, f_k\rangle_{\mathcal{E}} = \lim_{k\to\infty}\langle \varphi, \Delta f_k\rangle_{\mathcal{E}} = \lim_{k\to\infty}\langle \varphi, \delta_x - \varepsilon_k f_k\rangle_{\mathcal{E}} = \langle \varphi, \delta_x\rangle_{\mathcal{E}},$$

so that w is a monopole at x. $\square$

Lemma 4.8. *On any network, $(\text{ran}\,\Delta_{\mathcal{M}})^{clo} \subseteq \mathcal{F}in$ and hence $\mathcal{H}arm \subseteq \ker\Delta_{\mathcal{M}}^*$.*

Proof. If $v \in \mathcal{M}$, then clearly $\Delta_{\mathcal{M}}v \in \mathcal{F}in$. To close the operator, we consider sequences $\{u_n\} \subseteq \mathcal{M}$ which are Cauchy in $\mathcal{E}$, and for which $\{\Delta u_n\}$ is also Cauchy in $\mathcal{E}$, and then include $u := \lim u_n$ in $\text{dom}\,\Delta_{\mathcal{M}}$ by defining $\Delta_{\mathcal{M}}u := \lim \Delta_{\mathcal{M}}u_n$. Since $f_n := \Delta_{\mathcal{M}}u_n$ has finite support for each n, the $\mathcal{E}$-limit of $\{f_n\}$ must lie in $\mathcal{F}in$. Since $\mathcal{F}in$ is closed, the first claim follows. The second claim follows upon taking orthogonal complements. $\square$

Theorem 4.9. *The network is transient if and only if $(\text{ran}\,\Delta_{\mathcal{M}}^*)^{c\ell} = \mathcal{F}in$.*

Proof. ($\Rightarrow$) If the network is transient, we have a monopole at every vertex; see Remark 3.2. Then any $u \in \text{span}\{\delta_x\}$ is in $\text{ran}\,\Delta_{\mathcal{M}}^*$ because the monopole w_x is in $\text{dom}\,\Delta_{\mathcal{M}}$, and so $\mathcal{F}in \subseteq \text{ran}\,\Delta_{\mathcal{M}}^*$. The other inclusion is Lemma 4.8.

($\Leftarrow$) If $\delta_x \in \text{ran}\,\Delta_{\mathcal{M}}$ for some $x \in G^0$, then $\Delta_{\mathcal{M}}w = \delta_x$ for $w \in \text{dom}\,\Delta_{\mathcal{M}} \subseteq \text{dom}\,\mathcal{E}$ and so w is a monopole. Then the induced current dw is a unit flow to infinity, and the network is transient, again by [Lyo83]. $\qquad\square$

4.1. **Comparison with the grounded energy space.** There are some subtleties in the relationship between $\mathcal{H}_{\mathcal{E}}$ and **D** as discussed in [LP09] and [KY89, KY84, MYY94, Yam79], so we take a moment to give details. We have attempted to match the notation of these sources.

Definition 4.10. The inner product

$$\langle u, v \rangle_o := \overline{u(o)}v(o) + \langle u, v \rangle_{\mathcal{E}}.$$

makes $\text{dom}\,\mathcal{E}$ into a Hilbert space **D** which we call the *grounded energy space*. Let $\mathbf{D}_0$ be the closure of $\text{span}\{\delta_x\}$ in **D** and let **HD** be the space of harmonic functions in **D**.

Throughout this section (only), we use the notation $u_o := u(o)$, for $u \in \mathbf{D}$.

Definition 4.11. With regard to **D**, we define the vector subspace

$$\mathcal{M}_o^- := \{u \in \mathbf{D} : \Delta u = -u_o \delta_o\}. \tag{4.3}$$

Note that $\mathcal{M}_o^-$ contains the harmonic subspace

$$\mathbf{HD}_o := \{u \in \mathbf{D} : \Delta u = 0 \text{ and } u_o = 0\}. \tag{4.4}$$

The previous definition is motivated by the following lemma.

Lemma 4.12. $\mathbf{D}_0^{\perp} = \mathcal{M}_o^-$ *and hence* $\mathbf{D} = \mathbf{D}_0 \oplus \mathcal{M}_o^-$.

Proof. With $u_o := u(o)$, we have $u \in \mathbf{D}_0^{\perp}$ iff $u \perp \text{span}\{\delta_x\}$, which means that

$$0 = \langle u, \delta_x \rangle_o = u_o \delta_x(o) + \langle u, \delta_x \rangle_{\mathcal{E}} = u_o \delta_{xo} + \Delta u(x), \qquad \forall x \in G^0, \tag{4.5}$$

which means $\Delta u = -u_o \delta_o$. $\qquad\square$

Let us denote the projection of **D** to $\mathbf{D}_0$ by $P_{\mathbf{D}_0}$ and the projection to $\mathbf{D}_0^{\perp}$ by $P_{\mathbf{D}_0}^{\perp}$.

Remark 4.13. The constant function **1** decomposes into a linear combination of two monopoles: let $v = P_{\mathbf{D}_0}\mathbf{1}$ and $u = P_{\mathbf{D}_0}^{\perp}\mathbf{1} = \mathbf{1} - v$, and observe that $\Delta u = -u_o \delta_o$ by Lemma 4.12 and that $\Delta v = \Delta(\mathbf{1} - u) = -\Delta u = u_o \delta_o$, so $u_o = 1 - v_o$ gives $\Delta v = (1 - v_o)\delta_o$. In general, the constant function $k\mathbf{1}$ decomposes into $v = P_{\mathbf{D}_0}k\mathbf{1}$ and $u = P_{\mathbf{D}_0}^{\perp}k\mathbf{1}$, where

$$\Delta v = (k - u_o)\delta_o \quad \text{and} \quad \Delta u = -u_o \delta_o.$$

With respect to the decomposition $\mathbf{D} = \mathbf{D}_0 \oplus \mathcal{M}_o^-$, given by Lemma 4.12, there are two monopoles $w_o^{(1)} \in \mathbf{D}_0$ and $w_o^{(2)} \in \mathcal{M}_o^-$ (which may be equal) such that $\mathbf{1} = u_o w_o^{(1)} - u_o w_o^{(2)}$. When one passes from **D** to $\mathcal{H}_{\mathcal{E}}$ by modding out constants, these components of **1** add together to form (possibly constant) harmonic functions. An example of this is given in Example 6.8.

Consequently, Lemma 4.12 yields a short proof of [LP09, Exc. 9.6c]: Prove that the
network is recurrent iff $\mathbf{1} \in \mathbf{D}_0$. To see this, observe that if u is the projection of $\mathbf{1}$ to
$\mathbf{D}_0^\perp$, then $u \neq 0$ iff there is a monopole. This result first appeared (in more general form)
in [Yam77, Thm. 3.2].

Remark 4.14. Despite the fact that Theorem 2.16 gives $\mathcal{H}_{\mathcal{E}} = \mathcal{F}in \oplus \mathcal{H}arm$, note that
$\mathbf{D} \neq \mathbf{D}_0 \oplus \mathbf{HD}$. This is a bit surprising, since $\mathcal{H}_{\mathcal{E}} = \mathbf{D}/\mathbb{C}\mathbf{1}$, etc., and this mistake has been
made in the literature, e.g. [Yam79, Thm. 4.1]. The discrepancy results from the way that
$\mathbf{1}$ behaves with respect to $P_{\mathbf{D}_0}$; this is easiest to see by considering

$$\mathbf{D}_0 + k := \{f + k\mathbf{1} : f \in \mathbf{D}_0, k \in \mathbb{C}\}, \qquad k \neq 0.$$

If the network is transient and $f \in \mathbf{D}_0 + k$, $k \neq 0$, then $f = g + k\mathbf{1}$ for some $g \in \mathbf{D}_0$, and

$$f = (g + kP_{\mathbf{D}_0}\mathbf{1}) + kP_{\mathbf{D}_0}^\perp\mathbf{1}$$

shows $f \notin \mathbf{D}_0$. Nonetheless, it is easy to check that $\mathbf{D}_0 + k$ is equal to the o-closure of
span $\delta_x + k$, and hence that $(\mathbf{D}_0 + \mathbb{C}\mathbf{1})/\mathbb{C}\mathbf{1} = \mathcal{F}in$. This appears in [LP09, Exc. 9.6b].
Similarly, note that for a general $h \in \mathbf{HD}$, one has $h = P_{\mathbf{D}_0}^\perp h + k\mathbf{1}$, so that $h \notin \mathbf{D}_0^\perp$.

We conclude with a curious lemma that can greatly simplify the computation of monopoles
of the form $P_{\mathbf{D}_0}\mathbf{1}$; it is used in Example 6.8. In the next lemma, $u_o = u(o)$, as above.

Lemma 4.15. *Let* $u \in \mathbf{D}_0^\perp$. *Then* $u = P_{\mathbf{D}_0}^\perp\mathbf{1}$ *if and only if* $u_o = \mathcal{E}(u) + u_o^2 \in [0, 1)$.

Proof. From $\|u\|_o^2 + \|\mathbf{1} - u\|_o^2 = \|\mathbf{1}\|_o^2 = 1$, one obtains $\mathcal{E}(u) - u_o + |u_o|^2 = 0$. From $\langle u, \mathbf{1} - u\rangle_o = 0$, one obtains $\mathcal{E}(u) - \overline{u_o} + |u_o|^2 = 0$. Combining the equations gives $u_o = u_o = \frac{1}{2}(1 \pm \sqrt{1 - 4\mathcal{E}(u)})$, so that $u_o \in [0, 1]$. However, $u_o \neq 1$ or else $\mathcal{E}(u) = 0$ would imply $\mathbf{1} \in \mathbf{D}_0^\perp$ in contradiction to (4.5). The converse is clear.   □

Remark 4.16. The significance of the parameter u_o is not clear. However, it appears to
be related to the overall "strength" of the conductance of the network; we will see in
Example 6.8 that $u_o \approx 1$ corresponds to rapid growth of c near ∞. Also, it follows from the
Remark 4.13 and Lemma 4.15 that $u_o = 0$ corresponds to the recurrence. There is probably
a good interpretation of u_o in terms of probability and/or the speed of the random walk, but
we have not yet determined it. The existence of conductances attaining maximal energy
$\mathcal{E}(P_{\mathbf{D}_0}^\perp\mathbf{1}) = \frac{1}{4}$ is similarly intriguing, and even more mysterious. Example 6.8 shows that
the maximum is attained on $(\mathbb{Z}, c^n)$ for $c = 2$.

5. Applications and extensions

In §5.1, we use the techniques developed above to obtain new and succinct proofs of
four known results, and in §5.2 we give some useful special cases of our main result,
Theorem 3.12.

Definition 5.1. For an infinite graph G, we say $u(x)$ *vanishes at* ∞ iff for any exhaustion
$\{G_k\}$, one can always find k and a constant C such that $\|u(x) - C\|_\infty < \varepsilon$ for all $x \notin G_k$.
One can always choose the representative of $u \in \mathcal{H}_{\mathcal{E}}$ so that $C = 0$, but this may not be
compatible with the choice $u(o) = 0$.

Definition 5.2. Say $\gamma = (x_0, x_1, x_2, \dots)$ is a *path to* ∞ iff $x_i \sim x_{i-1}$ for each i, and for any
exhaustion $\{G_k\}$ of G,

$$\forall k, \exists N \text{ such that } n \geq N \implies x_n \notin G_k. \tag{5.1}$$

5.1. **More about $\mathcal{F}in$ and $\mathcal{H}arm$.** The next two results are almost converse to each other, although the exact converse of Lemma 5.3 is false; see [JP09c, Fig. 10 or Ex. 14.16]. However, the converse does hold almost everywhere with respect to the usual (Kolmogorov) measure on the space of paths starting at o, by [ALP99, Cor. 1.2] (see [Car72] for construction of the measure). Lemma 5.3 is also related to [Soa94, Thm. 3.86], in which the result is stated as holding almost everywhere with respect to the notion of extremal length.

Lemma 5.3. *If $u \in \mathcal{H}_{\mathcal{E}}$ and u vanishes at ∞, then $u \in \mathcal{F}in$.*

Proof. Let $u = f + h \in \mathcal{H}_{\mathcal{E}}$ vanish at ∞. This implies that for any exhaustion $\{G_k\}$ and any $\varepsilon > 0$, there is a k and C for which $\|h(x) - C\|_\infty < \varepsilon$ outside G_k. A harmonic function can only obtain its maximum on the boundary, unless it is constant, so in particular, ε bounds $\|h(x) - C\|_\infty$ on all of G_k. Letting $\varepsilon \to 0$, h tends to a constant function and $u = f$. $\square$

Lemma 5.4. *If $h \in \mathcal{H}arm$ is nonconstant, then from any $x_0 \in G^0$, there is a path to infinity $\gamma = (x_0, x_1, \dots)$, with $h(x_j) < h(x_{j+1})$ for all $j = 0, 1, 2, \dots$.*

Proof. Abusing notation, let h be any representative of h. Since $h(x) = \sum_{y \sim x} \frac{c_{xy}}{c(x)} h(y) \leq \sup_{y \sim x} h(y)$ and h is nonconstant, we can always find $y \sim x$ for which $h(y_1) > h(x_0)$. This follows from the maximal principle for harmonic functions; cf. [LP09, §2.1], [LPW08, Ex. 1.12], or [Soa94, Thm. 1.35]. Thus, one can inductively construct a sequence which defines the desired path γ. Note that γ is infinite, so the condition $h(x_j) < h(x_{j+1})$ eventually forces it to leave any finite subset of G^0, so Definition 5.2 is satisfied. $\square$

It is instructive to prove the contrapositive of Lemma 5.3 directly:

Lemma 5.5. *If $h \in \mathcal{H}arm \setminus \{0\}$, then h has at least two different limiting values at ∞.*

Proof. Choose $x \in G^0$ for which $h_x = P_{\mathcal{H}arm} v_x \in \mathcal{H}_{\mathcal{E}}$ is nonconstant. Then Lemma 5.4 gives a path to infinity γ_1 along which h_x is strictly increasing. Since the reasoning of Lemma 5.4 works just as well with the inequalities reversed, we also get γ_2 to ∞ along which h_x is strictly decreasing. This gives two different limiting values of h_x, and hence h_x cannot vanish at ∞. $\square$

Corollary 5.6. *If $h \in \mathcal{H}arm$ is nonconstant, then $h \notin \ell^p(G^0)$ for any $1 \leq p < \infty$.*

Proof. Lemma 5.5 shows that no matter what representative is chosen for h, the sum $\|h\|_p = \sum_{x \in G^0} |h(x)|^p$ has the lower bound $\sum_{x \in F} \varepsilon^p = \varepsilon^p |F|$, for some infinite set $F \subseteq G^0$. $\square$

5.2. **Special cases of the Discrete Gauss-Green formula.** In this subsection, we establish that the boundary term of Theorem 3.12 vanishes for vectors in $\text{span}\{v_x\}$ in Lemma 5.8, and that Δ is Hermitian when its domain is correctly chosen, in Corollary 5.10. These results continue the theme of Theorem 4.6.

Lemma 5.7. *For $u \in \text{span}\{v_x\}$, $\sum_{x \in G^0} \Delta u(x) = 0$.*

Proof. For a finite sum $u = \sum a_y v_y$, the result follows by interchanging finite sums:

$$\sum_{x \in G^0} \Delta v_y(x) = \sum_x a_y (\delta_y - \delta_o)(x) = \sum a_y - \sum a_y = 0. \qquad \square$$

Physically, $\sum \Delta u$ is the net divergence of the current passing through the network. Thus, Lemma 5.7 can be rephrased as saying that elements of span$\{v_x\}$ are "balanced"; compare to [Soa94, p. 45]. Lemma 5.7 is false for $u = w_x \in \mathcal{M}_x$ and may also fail for u in the closure of span$\{v_x\}$ (with respect to $\mathcal{E}$ or the graph norm of Δ).

Lemma 5.8. *If $v \in$ span$\{v_x\}$, then $\langle u, v \rangle_{\mathcal{E}} = \sum_{x \in G^0} \overline{u(x)} \Delta v(x)$.*

Proof. It suffices to consider $v = v_x$, whence

$$\sum_{G^0} u(y) \Delta v_x(y) = \sum_{G^0} u(y)(\delta_x - \delta_o)(y) = u(x) - u(o) = \langle u, v_x \rangle_{\mathcal{E}},$$

by Lemma 2.14 and the reproducing property of Corollary 2.7. □

Note that the formula in Lemma 5.8 may look odd because the right-hand side appears to depend on a choice of representatives, but this not the case by Lemma 5.7. However, Lemma 5.8 *is* false for $v \in \mathcal{M} \setminus$ span$\{v_x\}$.

When $\deg(x) < \infty$ for all $x \in G^0$, the next result follows immediately from Lemma 5.8 and Lemma 2.21. However, it is true even when there are vertices of infinite degree.

Theorem 5.9. *For $u, v \in$ span$\{v_x\}$,*

$$\langle u, \Delta v \rangle_{\mathcal{E}} = \sum_{x \in G^0} \overline{\Delta u(x)} \Delta v(x). \tag{5.2}$$

Proof. Let $u \in$ span$\{v_x\}$ be given by the finite sum $u = \sum_x \xi_x v_x$. We may assume the sum does not include o (see Definition 2.6). Then

$$\Delta u(y) = \sum_x \xi_x \Delta v_x(y) = \sum_x \xi_x (\delta_x - \delta_o)(y) = \xi_y. \tag{5.3}$$

Now we have

$$\langle u, \Delta u \rangle_{\mathcal{E}} = \sum_{x,y} \overline{\xi_x} \xi_y \langle v_x, \Delta v_y \rangle_{\mathcal{E}} = \sum_{x,y} \overline{\xi_x} \xi_y \langle v_x, \delta_y - \delta_o \rangle_{\mathcal{E}}.$$

Since it is easy to compute $\langle v_x, \delta_y - \delta_o \rangle_{\mathcal{E}} = \delta_{xy} + 1$ (Kronecker's delta), we have

$$\langle u, \Delta u \rangle_{\mathcal{E}} = \sum_{x,y} \overline{\xi_x} \xi_y (\delta_{xy} + 1) = \sum_x |\xi_x|^2 + \left| \sum_x \xi_x \right|^2 = \sum_x |\Delta u(x)|^2 + \left| \sum_x \Delta u(x) \right|^2, \tag{5.4}$$

by (5.3). The second sum vanishes by Lemma 5.7, and (5.2) follows by polarizing. □

Corollary 5.10. $\Delta_{\mathcal{M}}$ *satisfies*

$$0 \leq \sum_{x \in G^0} |\Delta u(x)|^2 \leq \langle u, \Delta u \rangle_{\mathcal{E}} < \infty. \tag{5.5}$$

Proof. For $u \in \mathcal{M}$, we can repeat the proof of Theorem 5.9 up to (5.4) to obtain the central inequality of (5.5); the inequality appears because the second sum may not vanish in this case. Now let $u \in$ dom $\Delta_{\mathcal{M}}$ and choose $\{u_n\} \subseteq \mathcal{M}$ with $\lim_{n \to \infty} \|u_n - u\|_{\mathcal{E}} = \lim_{n \to \infty} \|\Delta u_n - \Delta u\|_{\mathcal{E}} = 0$. Then the result follows by Fatou's lemma as in [Mal95, Thm. I.7.7]. □

Remark 5.11. The notation $u \in \ell^1$ means $\sum_{x \in G^0} |u(x)| < \infty$ and the notation $u \in \ell^2$ means $\sum_{x \in G^0} |u(x)|^2 < \infty$. When discussing an element u of $\mathcal{H}_\mathcal{E}$, we say u lies in ℓ^2 if it has a representative which does, i.e., if $u + k \in \ell^2$ for some $k \in \mathbb{C}$. This constant is clearly necessarily unique on an infinite network, if it exists.

The next result extends Proposition 1.9 and is a partial converse to Theorem 3.12. All that is required for the computation in the proof of Lemma 5.12 is that $u\Delta v \in \ell^1$, which is certainly implied by $u, \Delta v \in \ell^2$. However, this would not suffice to show $u, v \in \mathrm{dom}\,\mathcal{E}$.

Lemma 5.12. *If* $u, v, \Delta u, \Delta v \in \ell^2$*, then* $\langle u, v \rangle_\mathcal{E} = \sum_{x \in G^0} u(x)\Delta v(x)$*, and* $u, v \in \mathrm{dom}\,\mathcal{E}$.

Proof. If $u, \Delta v \in \ell^2$, then $u\Delta v \in \ell^1$, and the proof of Proposition 1.9 is still valid: the absolute convergence of $\sum_{x \in G^0} u(x)\Delta v(x)$ justifies the rearrangement. Substituting u for v in this formula gives $u \in \mathrm{dom}\,\mathcal{E}$, and similarly for v. □

Lemma 5.13. *Let* $w_z \in \mathcal{H}_\mathcal{E}$ *be a monopole at* z*. Then* $w_z \in \mathrm{ran}(\mathbb{I} + \Delta_\mathcal{M})$ *if and only if there is a function* $u \in \mathcal{H}_\mathcal{E}$ *satisfying* $u + \Delta u = 0$ *on* $G^0 \setminus \{z\}$.

Proof. ($\Rightarrow$) Since $w_z = v + \Delta v$ for some $v \in \mathcal{H}_\mathcal{E}$, set $u := v - w_z$. Then for $x \neq z$, it is easy to check $u(x) + \Delta u(x) = 0$. ($\Leftarrow$) Set $v := w_z + au$ for $a := -1/(u(z) + \Delta u(z))$. Then it is easy to check that $v(x) + \Delta v(x) = w_z(x)$. □

If there are vertices of infinite degree in the network, then it does not necessary follow that $\mathrm{span}\{\delta_x\} \subseteq \mathrm{span}\{v_x\}$, or that $\mathrm{span}\{\delta_x\} \subseteq \mathcal{M}$. However, we do have the following result.

Definition 5.14. Let $\mathcal{F} := \mathrm{span}\{\delta_x\}_{x \in G^0}$ denote the vector space of *finite* linear combinations of Dirac masses, and let $\Delta_\mathcal{F}$ be the closure of the Laplacian when taken to have the domain $\mathcal{F}$.

Note that $\mathcal{F}$ is a dense domain only when $\mathcal{H}arm = 0$, by Corollary 2.17. Again, since Δ agrees with $\Delta_\mathcal{F}$ pointwise, we may suppress reference to the domain for ease of notation. The next result extends Proposition 1.9 to networks with vertices of infinite degree.

Theorem 5.15. *If* u *or* v *lies in* $\mathrm{dom}\,\Delta_\mathcal{F}$*, then* $\langle u, v \rangle_\mathcal{E} = \sum_{x \in G^0} \overline{u(x)}\Delta v(x)$.

Proof. First, suppose $u \in \mathrm{dom}\,\Delta_\mathcal{F}$ and choose a sequence $\{u_n\} \subseteq \mathrm{span}\{\delta_x\}$ with $\|u_n - u\|_\mathcal{E} \to 0$. From Lemma 2.12, one has $\langle \delta_x, v \rangle_\mathcal{E} = \Delta v(x)$, and hence $\langle u_n, v \rangle_\mathcal{E} = \sum_{x \in G^0} u_n(x)\Delta v(x)$ holds for each n. Define $M := \sup\{\|u_n\|_\mathcal{E}\}$, and note that $M < \infty$, since this sequence is convergent (to $\|u\|_\mathcal{E}$). Moreover, $|\langle u_n, v \rangle_\mathcal{E}| \leq M \cdot \|v\|_\mathcal{E}$ by the Schwarz inequality. Since u_n converges pointwise to u in $\mathcal{H}_\mathcal{E}$ by Lemma 2.25, this bound will allow us to apply Fatou's Lemma (as stated in [Mal95, Lemma 7.7], for example), as follows:

$$\langle u, v \rangle_\mathcal{E} = \lim_{n \to \infty} \langle u_n, v \rangle_\mathcal{E} = \lim_{n \to \infty} \sum_{x \in G^0} \overline{u_n(x)}\Delta v(x) = \sum_{x \in G^0} \overline{u(x)}\Delta v(x).$$

where we have used the hypothesis followed by $u_n \in \mathrm{span}\{\delta_x\}$. Note that the sum over G^0 is absolutely convergent, as required by Definition 1.5.

Now suppose that $v \in \mathrm{dom}\,\Delta_\mathcal{F}$ and observe that this implies $v \in \mathcal{F}in$ also. By Theorem 2.16, one can decompose $u = f + h$ where $f = P_{\mathcal{F}in}u$ and $h = P_{\mathcal{H}arm}u$, and then

$$\langle u, v \rangle_\mathcal{E} = \langle f, v \rangle_\mathcal{E} + \langle h, v \rangle_\mathcal{E} = \langle f, v \rangle_\mathcal{E},$$

since h is orthogonal to v. Now apply the previous argument to $\langle f, v \rangle_\mathcal{E}$. □

For networks of finite degree, Theorem 5.15 follows from Theorem 3.12 by Lemma 2.21.

<center>6. EXAMPLES</center>   1

In this section, we introduce the most basic family of examples that illustrate our tech-   2
nical results and exhibit the properties (and support the types of functions) that we have   3
discussed above. After presenting some basic examples, we prove some theorems regard-   4
ing the properties of these examples.   5

Networks similar to Example 6.1 have been discussed elsewhere in the literature (for   6
example, [KY89, Ex. 3.12, Ex. 4.9] and [KY84, Ex. 3.1, Ex. 3.2]), but the authors appear   7
to assume that Δ is self-adjoint. This is not generally the case when c is unbounded;   8
in fact, the Laplacian is *not* self-adjoint for Example 6.3 or Example 6.2. The proof is   9
unfortunately beyond the scope of this paper; see [JP09c, Ex. 14.36 and Ex. 14.39] for   10
further discussion and the explicit computation of defect vectors.   11

Example 6.1 (Integer networks). Let $(\mathbb{Z}, c)$ denote the network with integers for vertices,   12
and with conductances defined by c. We fix $o = 0$.   13

These networks are more interesting when c grows fast enough to ensure, for example,   14
that $\sum c_{xy}^{-1} < \infty$. In this case, it is helpful to keep the following more concrete model in   15
mind, especially if one hopes for tractable computations.   16

Example 6.2 (Geometric integer model). For a fixed constant $c > 1$, let $(\mathbb{Z}, c^n)$ denote the   17
network with integers for vertices, and with geometrically increasing conductances defined   18
by $c_{n-1,n} = c^{\max\{|n|,|n-1|\}}$ so that the network under consideration is   19

$$\cdots \xrightarrow{c^3} -2 \xrightarrow{c^2} -1 \xrightarrow{c} 0 \xrightarrow{c} 1 \xrightarrow{c^2} 2 \xrightarrow{c^3} 3 \xrightarrow{c^4} \cdots$$

Again, we fix $o = 0$. Theorem 6.4 shows that $\mathcal{H}arm \neq 0$ for this network, and Lemma 6.5   20
exhibits dipoles, monopoles, and a harmonic function on Example 6.2.   21

Example 6.3 (Geometric half-integer model). It is also interesting to consider $(\mathbb{Z}_+, c^n)$, as   22
this network supports a monopole, but has $\mathcal{H}arm = 0$.   23

$$0 \xrightarrow{c} 1 \xrightarrow{c^2} 2 \xrightarrow{c^3} 3 \xrightarrow{c^4} \cdots$$

Lemma 6.5 exhibits dipoles and a monopole for this example, but this network does not   24
support harmonic functions.   25

For $k = 2, 3, \ldots$, the network $(\mathbb{Z}_+, k^n)$ can be thought of as the "projection" of the   26
homogeneous tree of degree k $(\mathcal{T}_k, \frac{1}{k}\mathbf{1})$ under a map which sends x to $n \in \mathbb{Z}$ iff there are n 27
edges between x and o.   28

Theorem 6.4. $\mathcal{H}arm \neq 0$ *for* $(\mathbb{Z}, c)$ *iff* $\sum c_{xy}^{-1} < \infty$. *In this case,* $\mathcal{H}arm$ *is spanned by a*   29
single bounded function.   30

Proof. $(\Rightarrow)$ Fix $u(0) = 0$, define $u(1) = \frac{1}{c_{01}}$ and let $u(n)$ be such that   31

$$u(n) - u(n-1) = \frac{1}{c_{n-1,n}}, \quad \forall n. \tag{6.1}$$

Now u is harmonic: 32

$$\Delta u(n) = c_{n-1,n}(u(n) - u(n-1)) - c_{n,n+1}(u(n+1) - u(n))$$

$$= c_{n-1,n} \frac{1}{c_{n-1,n}} - c_{n,n+1} \frac{1}{c_{n,n+1}} = 0,$$

and u is of finite energy

$$\mathcal{E}(u) = \sum_{n \in \mathbb{Z}} c_{n-1,n}(u(n) - u(n-1))^2 = \sum_{n \in \mathbb{Z}} \frac{1}{c_{n-1,n}} < \infty.$$

Note that once the values of $u(0)$ and $u(1)$ are fixed, all the other values of $u(n)$ are determined by (6.1). Therefore, $\mathcal{H}arm$ is 1-dimensional.

($\Leftarrow$) If $\Delta u(n) = c_{n-1,n}(u(n) - u(n-1)) - c_{n,n+1}(u(n+1) - u(n)) = 0$ for every n, then

$$c_{n-1,n}(u(n) - u(n-1)) = c_{n,n+1}(u(n+1) - u(n)) = a,$$

for some fixed a (the amperage of a sourceless current). Then

$$\mathcal{E}(u) = \sum_{n \in \mathbb{Z}} c_{n-1,n}(u(n) - u(n-1))^2 = a^2 \sum_{n \in \mathbb{Z}} \frac{1}{c_{n-1,n}} < \infty, \qquad (6.2)$$

since $u \in \mathcal{H}arm \subseteq \mathcal{H}_{\mathcal{E}}$. Note that (6.2) implies u is bounded: $\mathcal{E}(u) = a \sum_{n \in \mathbb{Z}}(u(n) - u(n-1))$ and $\sum_{n \geq 1}(u(n) - u(n-1)) = \lim_{n \to \infty} u(n) - u(0)$. The function u is monotonic because it is harmonic, so the sum is absolutely convergent. $\qquad \square$

Lemma 6.5. *On $(\mathbb{Z}, c^n)$, the energy kernel is given by*

$$v_n(k) = \begin{cases} 0, & k \leq 0, \\ \frac{1-r^{k+1}}{1-r}, & 1 \leq k \leq n, \, n > 0, \\ \frac{1-r^{n+1}}{1-r}, & k \geq n, \end{cases}$$

and similarly for $n < 0$. Furthermore, the function $w_o(n) = ar^{|n|}$, $a := \frac{r}{2(1-r)}$, defines a monopole, and $h(n) = \text{sgn}(n)(1 - w_o(n))$ defines an element of $\mathcal{H}arm$.

Proof. It is easy to check that $\Delta w_o(0) = 2c(a - ar) = 1$, and that $\Delta w_o(n) = c^n(ar^n - ar^{n-1}) + c^{n+1}(ar^n - ar^{n+1}) = 0$ for $n \neq 0$. The reader may check that $\mathcal{E}(w_o) = \frac{r}{2(1-r)}$ so that $w_o \in \mathcal{H}_{\mathcal{E}}$. The computations for v_x and h are essentially the same. $\qquad \square$

Remark 6.6. With $h(n) = \text{sgn}(n)(1 - w_o(n))$ defined as in Lemma 6.5, the boundary term is 1. To compute this, use the exhaustion $G_k = [-k, k]$,

$$\sum_{x \in \text{bd } G_k} h(x) \frac{\partial h}{\partial \mathrm{n}}(x) = 2c^k(ar^k - ar^{k-1}) = 2a(r-1)\frac{1}{r} = 1,$$

so that $\sum_{\text{bd } G} h \frac{\partial h}{\partial \mathrm{n}} = \lim_{k \to \infty} \sum_{\text{bd } G_k} h \frac{\partial h}{\partial \mathrm{n}} = 1$.

Remark 6.7. As in Lemma 6.5, it is straightforward to check that $w_o(n) = ar^{|n|}$, $a := \frac{r}{(1-r)}$, defines a monopole on the geometric half-integer model $(\mathbb{Z}_+, c^n)$. However, it is also easy to check by induction that $\mathcal{H}arm = 0$ for this model.

Example 6.8 (Decomposition in **D**). In Remark 4.14, we discussed the Hilbert space **D** and its inner product $\langle u, v \rangle_o := u(o)v(o) + \langle u, v \rangle_{\mathcal{E}}$. Since $(\mathbb{Z}_+, c^n)$ and $(\mathbb{Z}, c^n)$ are both transient for $c > 1$ (but only the latter contains harmonic functions), it is interesting to consider $P_{\mathbf{D}_0}\mathbf{1}$ for these models (see Remark 4.13). The projections $v = P_{\mathbf{D}_0}\mathbf{1}$ and $u = \mathbf{1} - v = P_{\mathbf{D}_0}^{\perp}\mathbf{1}$ on $(\mathbb{Z}, c^n)$ are given by

$$v(x) = 2 - 2a + ar^{|x|} \qquad \text{and} \qquad u(x) = 2a - 1 - ar^{|x|}, \tag{6.3}$$

where with $a = \frac{1}{3-2c}$ and $r = c^{-1}$, and one can check $u \in \mathcal{M}_o^-$; see Definition 4.11 and   1
Lemma 4.12. In particular, $\Delta v = (1 - v_o)\delta_0$ and $\Delta u = -u_o\delta_0$ (as usual, $o = 0$). Now   2
consider the representative of $w \in \mathcal{M}_o$ given by   3

$$w(x) = (2 - a)\mathcal{X}_{[-\infty, 0]} + \left(1 + 2a(r^{|x|} - c)\right)\mathcal{X}_{[1, \infty)}. \tag{6.4}$$

A straightforward computation shows that $w = v + h$ with $h \in \mathbf{HD}_o$.   4
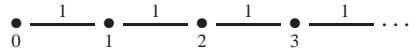
The function $v = P_{\mathbf{D}_0}\mathbf{1}$ was computed for $(\mathbb{Z}, c^n)$ in (6.3) by using the formula $\mathcal{E}(u) =$ 5
$u_o - u_o^2$, from Lemma 4.15, where $u := P_{\mathbf{D}_0}^\perp \mathbf{1} = \mathbf{1} - v$ and $u_o = u(o)$. For a general   6
network (G, c), this formula implies that $(u_o, \mathcal{E}(u))$ lies on a parabola with $u_o \in [0, 1)$ and   7
maximum at $(\frac{1}{2}, \frac{1}{4})$. From (6.3), it is clear that the network $(\mathbb{Z}, c^n)$ provides an example of   8
how $u_o = 1 - \frac{1}{2c-1}$ can take any value in $[0, 1)$. Note that $c = 1$ corresponds to $\mathcal{E}(u) = 0$,   9
which is the recurrent case.   10

Example 6.9 (Star networks). Let $(\mathcal{S}_m, c^n)$ be a network constructed by conjoining m 11
copies of $(\mathbb{Z}_+, c^n)$ by identifying the origins of each; let o be the common origin.   12

In [JP09a], we explore the boundary $\text{bd}\, G$ in more detail. The idea is that the boundary   13
term is nontrivial precisely when $\text{bd}\, G \neq \varnothing$. The presence of a monopole indicates that   14
$\text{bd}\, G$ contains at least one point; see Theorem 4.6. If $\mathcal{H}arm \neq 0$, then there are at least two   15
boundary points; see Lemma 5.5 and Corollary 4.4.   16

Example 6.9 shows how to construct a network which has a boundary with cardinality   17
m. Note that these boundary points can be distinguished by monopoles as in (6.4). This   18
monopole acts as a sort of indicator function for the corresponding boundary point $+\infty$;   19
when we pass to $\mathcal{H}_{\mathcal{E}}$, $\tilde{w}$ is supported only on the positive half of $\mathbb{Z}$. We return to the   20
general case (G, c) in [JP09a].   21

Example 6.10 (Independence of exhaustion). The following elementary example shows   22
how the boundary term may not be independent of exhaustion if one does not restrict to   23
$u \in \mathcal{M}$; this example is adapted from [KY89, Ex. 3.12]. Take the nonnegative integers   24
$\{0, 1, 2, 3, \dots\}$ with unit conductance edges connecting nearest neighbours, as depicted.   25

$$\bullet \!\!\overset{1}{\rule{1cm}{0.4pt}}\!\! \bullet \!\!\overset{1}{\rule{1cm}{0.4pt}}\!\! \bullet \!\!\overset{1}{\rule{1cm}{0.4pt}}\!\! \bullet \!\!\overset{1}{\rule{1cm}{0.4pt}}\!\! \dots$$
$$0 \qquad\quad 1 \qquad\quad 2 \qquad\quad 3$$

Define a function u on the vertices by   26

$$u(0) = 0 \qquad \text{and} \qquad u(n) = u(n - 1) + \begin{cases} 1/k, & \text{if } n = 2^k, \\ 1/n, & \text{otherwise,} \end{cases}$$

so that the increment between neighbours is either $\frac{1}{n}$ or $\frac{1}{k} = \frac{1}{\log_2 n}$. The energy is estimated 27

$$\mathcal{E}(u) \leq \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$

which proves that $u \in \mathcal{H}_{\mathcal{E}}$. Note, however, that u is unbounded, so that $u \notin \mathcal{M}$. Let $\{G_k\}_{k=1}^{\infty}$ 28
be an exhaustion by the sets $G_k = [0, 2^k]$, so that $\text{bd}\, G_k = \{2^k\}$. Then the boundary sum   29
contains only one term and we have   30

$$\sum_{\text{bd}\,G} u(x)\tfrac{\partial u}{\partial \mathbb{n}}(x) = u(2^k)\tfrac{\partial u}{\partial \mathbb{n}}(2^k) \geq \left(\frac{1}{2}\sum_{j=1}^{2^k}\frac{1}{j}\right)\cdot\frac{1}{k} \geq \log 2.$$

On the other hand, let $\{G'_k\}_{k=1}^{\infty}$ be an exhaustion by the sets $G'_k = [0, 3^k]$, so that $\text{bd}\,G'_k = \{3^k\}$. Then the boundary sum still contains only one term but now we have

$$\sum_{\text{bd}\,G} u(x)\tfrac{\partial u}{\partial \mathbb{n}}(x) = u(3^k)\tfrac{\partial u}{\partial \mathbb{n}}(3^k) \leq \left(\sum_{j=1}^{3^k}\frac{1}{j}\right)\cdot\frac{1}{3^k} \xrightarrow{\ n\to\infty\ } 0.$$

REFERENCES

[AAL08] Daniel Alpay, Haim Attia, and David Levanony. Une généralisation de l'intégrale stochastique de Wick-Itô. *C. R. Math. Acad. Sci. Paris*, 346(5-6):261–265, 2008.

[AD06] D. Alpay and C. Dubi. Some remarks on the smoothing problem in a reproducing kernel Hilbert space. *J. Anal. Appl.*, 4(2):119–132, 2006.

[ADV09] Daniel Alpay, Aad Dijksma, and Dan Volok. Schur multipliers and de Branges-Rovnyak spaces: the multiscale case. *J. Operator Theory*, 61(1):87–118, 2009.

[AL08] Daniel Alpay and David Levanony. On the reproducing kernel Hilbert spaces associated with the fractional and bi-fractional Brownian motions. *Potential Anal.*, 28(2):163–184, 2008.

[ALP99] Alano Ancona, Russell Lyons, and Yuval Peres. Crossing estimates and convergence of Dirichlet functions along random walk and diffusion paths. *Ann. Probab.*, 27(2):970–989, 1999.

[Aro50] N. Aronszajn. Theory of reproducing kernels. *Trans. Amer. Math. Soc.*, 68:337–404, 1950.

[BH91] Nicolas Bouleau and Francis Hirsch. *Dirichlet forms and analysis on Wiener space*, volume 14 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1991.

[Bre67] M. Brelot. *Lectures on potential theory*. Notes by K. N. Gowrisankaran and M. K. Venkatesha Murthy. Second edition, revised and enlarged with the help of S. Ramaswamy. Tata Institute of Fundamental Research Lectures on Mathematics, No. 19. Tata Institute of Fundamental Research, Bombay, 1967.

[BS96] Itai Benjamini and Oded Schramm. Harmonic functions on planar and almost planar graphs and manifolds, via circle packings. *Invent. Math.*, 126(3):565–587, 1996.

[BV03] Joseph A. Ball and Victor Vinnikov. Formal reproducing kernel Hilbert spaces: the commutative and noncommutative settings. In *Reproducing kernel spaces and applications*, volume 143 of *Oper. Theory Adv. Appl.*, pages 77–134. Birkhäuser, Basel, 2003.

[Car72] P. Cartier. Fonctions harmoniques sur un arbre. In *Symposia Mathematica, Vol. IX (Convegno di Calcolo delle Probabilità, INDAM, Rome, 1971)*, pages 203–270. Academic Press, London, 1972.

[Car73] P. Cartier. Harmonic analysis on trees. In *Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972)*, pages 419–424. Amer. Math. Soc., Providence, R.I., 1973.

[CC72] Corneliu Constantinescu and Aurel Cornea. *Potential theory on harmonic spaces*. Springer-Verlag, New York, 1972. With a preface by H. Bauer, Die Grundlehren der mathematischen Wissenschaften, Band 158.

[Chu01] Fan Chung. *Spectral Graph Theory*. Cambridge, 2001.

[CL07] Lincoln Chayes and Thomas M. Liggett. One dimensional nearest neighbor exclusion processes in inhomogeneous and random environments. *J. Stat. Phys.*, 129(2):193–203, 2007.

[CW92] Donald I. Cartwright and Wolfgang Woess. Infinite graphs with nonconstant Dirichlet finite harmonic functions. *SIAM J. Discrete Math.*, 5(3):380–385, 1992.

[DK88] Jozef Dodziuk and Leon Karp. Spectral and function theory for combinatorial Laplacians. In *Geometry of random motion (Ithaca, N.Y., 1987)*, volume 73 of *Contemp. Math.*, pages 25–40. Amer. Math. Soc., Providence, RI, 1988.

[Dod06] Józef Dodziuk. Elliptic operators on infinite graphs. In *Analysis, geometry and topology of elliptic operators*, pages 353–368. World Sci. Publ., Hackensack, NJ, 2006.

[Doy88] Peter G. Doyle. Electric currents in infinite networks. *Preprint*, pages 1–13, 1988. arXiv:math/0703899.

[DS84] Peter G. Doyle and J. Laurie Snell. *Random walks and electric networks*. Carus Monograph. Mathematical Association of America, 1984.

[DS88] Nelson Dunford and Jacob T. Schwartz. *Linear operators. Part II*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1988. Spectral theory. Selfadjoint operators in Hilbert space, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1963 original, A Wiley-Interscience Publication.

[DS99] Manfred Denker and Hiroshi Sato. Sierpiński gasket as a Martin boundary. II. The intrinsic metric. *Publ. Res. Inst. Math. Sci.*, 35(5):769–794, 1999.

[DS01] Manfred Denker and Hiroshi Sato. Sierpiński gasket as a Martin boundary. I. Martin kernels. *Potential Anal.*, 14(3):211–232, 2001.

[FHM03] M. Fannes, B. Haegeman, and M. Mosonyi. Entropy growth of shift-invariant states on a quantum spin chain. *J. Math. Phys.*, 44(12):6005–6019, 2003.

[FŌT94] Masatoshi Fukushima, Yōichi Ōshima, and Masayoshi Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1994.

[Han96] Kenji Handa. Entropy production per site in (nonreversible) spin-flip processes. *J. Statist. Phys.*, 83(3-4):555–571, 1996.

[JLW] Hongbing Ju, Ka-Sing Lau, and Xiang-Yang Wang. Post-critically finite fractals and martin boundary. *preprint*.

[JP09a] Palle E. T. Jorgensen and Erin P. J. Pearse. Boundaries of infinite resistance networks. *In review*, pages 1–24, 2009. arXiv:0906.2745.

[JP09b] Palle E. T. Jorgensen and Erin P. J. Pearse. A Hilbert space approach to effective resistance metrics. *To appear: Complex Anal. Oper. Theory*, pages 1–30, 2009. arXiv:0906.2535.

[JP09c] Palle E. T. Jorgensen and Erin P. J. Pearse. Operator theory of electrical resistance networks. pages 1–184, 2009. arXiv:0806.3881.

[JP09d] Palle E. T. Jorgensen and Erin P. J. Pearse. Resistance boundaries of infinite networks. *To appear:* In *Boundaries and Spectral Theory*, pages 1–32. Birkhauser, 2009. arXiv:0909.1518.

[JP09e] Palle E. T. Jorgensen and Erin P. J. Pearse. Spectral reciprocity and matrix representations of unbounded operators. *In preparation*, pages 1–34, 2009. arXiv:0911.0185.

[Kai03] Vadim A. Kaimanovich. Random walks on Sierpinski graphs: hyperbolicity and stochastic homogenization. In *Fractals in Graz 2001*, Trends Math., pages 145–183. Birkhäuser, Basel, 2003.

[Kat95] Tosio Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[Kig01] Jun Kigami. *Analysis on fractals*, volume 143 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001.

[Kig03] Jun Kigami. Harmonic analysis for resistance forms. *J. Funct. Anal.*, 204(2):399–444, 2003.

[Kig09a] Jun Kigami. Dirichlet forms and associated heat kernels on the cantor set induced by random walks on trees. 2009. *Preprint*.

[Kig09b] Jun Kigami. Resistance forms, quasisymmetric maps and heat kernel estimates. 2009. *Preprint*.

[KL10] Matthias Keller and Daniel Lenz. Dirichlet forms and stochastic completeness of graphs and subgraphs. *preprint*, page 17, 2010. arXiv:0904.2985.

[KY84] Takashi Kayano and Maretsugu Yamasaki. Boundary limit of discrete Dirichlet potentials. *Hiroshima Math. J.*, 14(2):401–406, 1984.

[KY89] Takashi Kayano and Maretsugu Yamasaki. Discrete Dirichlet integral formula. *Discrete Appl. Math.*, 22(1):53–68, 1988/89.

[Lig78] Thomas M. Liggett. Attractive nearest neighbor spin systems on the integers. *Ann. Probab.*, 6(4):629–636, 1978.

[LP09] Russell Lyons and Yuval Peres. *Probability on Trees and Graphs*. Unpublished (see Lyons' web site), 2009.

[LPW08] Daniel Levin, Yuval Peres, and Elizabeth Wilmer. *Markov Chains and Mixing Times*. American Mathematical Society, Providence, RI, 2008.

[LW09] Ka-Sing Lau and Xiang-Yang Wang. Self-similar sets as hyperbolic boundaries. *Indiana Univ. Math. J.*, 58(4):1777–1795, 2009.

[Lyo83] Terry Lyons. A simple criterion for transience of a reversible Markov chain. *Ann. Probab.*, 11(2):393–402, 1983.

[Mal95] Paul Malliavin. *Integration and probability*, volume 157 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With the collaboration of Hélène Airault, Leslie Kay and Gérard Letac, Edited and translated from the French by Kay, With a foreword by Mark Pinsky.

[MYY94] Atsushi Murakami, Maretsugu Yamasaki, and Yoshinori Yone-E. Some properties of reproducing kernels on an infinite network. *Mem. Fac. Sci. Shimane Univ.*, 28:1–8, 1994.

[NW59] C. St. J. A. Nash-Williams. Random walk and electric currents in networks. *Proc. Cambridge Philos. Soc.*, 55:181–194, 1959.

[PW88] Massimo A. Picardello and Wolfgang Woess. Harmonic functions and ends of graphs. *Proc. Edinburgh Math. Soc. (2)*, 31(3):457–461, 1988.

[PW90] M. A. Picardello and W. Woess. Ends of infinite graphs, potential theory and electrical networks. In *Cycles and rays (Montreal, PQ, 1987)*, volume 301 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 181–196. Kluwer Acad. Publ., Dordrecht, 1990.

[RSV97] Marco Rigoli, Maura Salvatori, and Marco Vignati. Subharmonic functions on graphs. *Israel J. Math.*, 99:1–27, 1997.

[RT10] Luke G. Rogers and Alexander Teplyaev. Laplacians on the basilica Julia sets. *Commun. Pure Appl. Anal.*, 9(1):211–231, 2010.

[Saw97] Stanley A. Sawyer. Martin boundaries and random walks. In *Harmonic functions on trees and buildings (New York, 1995)*, volume 206 of *Contemp. Math.*, pages 17–44. Amer. Math. Soc., Providence, RI, 1997.

[Sch38a] I. J. Schoenberg. Metric spaces and completely monotone functions. *Ann. of Math. (2)*, 39(4):811–841, 1938.

[Sch38b] I. J. Schoenberg. Metric spaces and positive definite functions. *Trans. Amer. Math. Soc.*, 44(3):522–536, 1938.

[Soa94] Paolo M. Soardi. *Potential theory on infinite networks*, volume 1590 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994.

[Str06] Robert S. Strichartz. *Differential equations on fractals*. Princeton University Press, Princeton, NJ, 2006. A tutorial.

[SW91] Paolo M. Soardi and Wolfgang Woess. Uniqueness of currents in infinite resistive networks. *Discrete Appl. Math.*, 31(1):37–49, 1991.

[Tho90] Carsten Thomassen. Resistances and currents in infinite electrical networks. *J. Combin. Theory Ser. B*, 49(1):87–102, 1990.

[vN32] J. von Neumann. Über adjungierte Funktionaloperatoren. *Ann. of Math. (2)*, 33(2):294–310, 1932.

[Web08] Andreas Weber. Analysis of the laplacian and the heat flow on a locally finite graph. page 17, 2008. arXiv:0801.0812.

[Woe86] Wolfgang Woess. Harmonic functions on infinite graphs. *Rend. Sem. Mat. Fis. Milano*, 56:51–63 (1988), 1986.

[Woe00] Wolfgang Woess. *Random walks on infinite graphs and groups*, volume 138 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2000.

[Woj07] Radosław K. Wojciechowski. Stochastic completeness of graphs. *Ph. D. Dissertation*, page 72, 2007. arXiv:0712.1570.

[Woj09] Radosław K. Wojciechowski. Heat kernel and essential spectrum of infinite graphs. *Indiana Univ. Math. J.*, 58(3):1419–1441, 2009. arXiv:0802.2745.

[Woj10] Radosław K. Wojciechowski. Stochastically incomplete manifolds and graphs. *preprint*, page 17, 2010. arXiv:0712.1570.

[Yam77] Maretsugu Yamasaki. Parabolic and hyperbolic infinite networks. *Hiroshima Math. J.*, 7(1):135–146, 1977.

[Yam79] Maretsugu Yamasaki. Discrete potentials on an infinite network. *Mem. Fac. Sci. Shimane Univ.*, 13:31–44, 1979.

[Zha09] Haizhang Zhang. Orthogonality from disjoint support in reproducing kernel Hilbert spaces. *J. Math. Anal. Appl.*, 349(1):201–210, 2009.

E-mail address: jorgen@math.uiowa.edu

UNIVERSITY OF IOWA, IOWA CITY, IA 52246-1419 USA

E-mail address: epearse@math.uiowa.edu

UNIVERSITY OF IOWA, IOWA CITY, IA 52246-1419 USA